# Second homology of generalized periplectic Lie superalgebras

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**Abstract**

Let $(R,^-)$ be an arbitrary unital associative superalgebra with superinvolution over a commutative ring $\Bbbk$ with 2 invertible. The second homology of the generalized periplectic Lie superalgebra $\mathfrak{p}_m(R,^-)$ for $m \geqslant 3$ has been completely determined via an explicit construction of its universal central extension. In particular, this second homology could be identified with the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $R$ with certain superinvolution whenever $m \geqslant 5$.

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## 1 Introduction

It is well known that the second homology of a Lie (super)algebra $\mathfrak{g}$ is identified with the kernel of its universal central extension, and thus classifies all central extension of $\mathfrak{g}$ up to isomorphism (c.f. [15, 17]). They play crucial rules in the theory of Lie (super)algebras.

A remarkable work about the second homology of a Lie algebra is the nice connection between the second homology of a matrix Lie algebra and the first cyclic homology of its coordinates associative algebra established in [10]. Concretely, let $A$ be a unital associative algebra over a commutative ring with 2 invertible, one denotes $\mathfrak{gl}_n(A)$ the Lie algebra of all $n \times n$-matrices with entries in $A$ under commutator operation and $\mathfrak{sl}_n(A)$ the derived Lie subalgebra of $\mathfrak{gl}_n(A)$. It is shown in [10] that the second homology $\mathrm{H}_2(\mathfrak{sl}_n(A))$ with $n \geqslant 2$ is isomorphic to the first cyclic homology $\mathrm{HC}_1(A)$.

Such an isomorphism has been extended to many other classes of Lie (super)algebras. For instance, Y. Gao showed in [6] that the second homology of elementary unitary Lie algebra $\mathfrak{eu}_n(R,^-)$ with $n \geqslant 5$ is identified with the first skew-dihedral homology of $(R,^-)$ that is a unital associative algebra with anti-involution. The super analogue of C. Kassel and J. L. Loday's work was obtained in [3, 4]. The isomorphism from the second homology of the Lie superalgebra $\mathfrak{sl}_{m|n}(S)$ coordinated by a unital associative superalgebra $S$ with $m+n \geqslant 5$ to the first $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology $\mathrm{HC}_1(S)$ was established. Recent investigation [2] further gave the identification between the second homology of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{m|2n}(R,^-)$ with positive integer pair $(m,n) \neq (1,1)$ or $(2,1)$ and the first $\mathbb{Z}/2\mathbb{Z}$-grade skew-dihedral homology of $(R,^-)$, where $(R,^-)$ is a unital associative superalgebra with superinvolution (see (2.3) for definition). A series of deeep investigations on the relationship between the homology theory of Lie algebras and the homology theory of associative algebras have been made in [13, 12].

Inspired by above developments, we aim to establish an isomorphism that is analogous to C. Kassel and J. L. Loday's isomorphism for the generalized periplectic Lie superalgebra $\mathfrak{p}_m(R,^-)$ coordinatized by a unital associative superalgebra $(R,^-)$ with superinvolution. As in Section 2,

a generalized periplectic Lie superalgebra is defined as the derived sub-supalgebra of the Lie superalgebra of all skew-symmetric matrices with respect to certain superinvolution. It could be understood as a super analogue of a unitary Lie algebra introduced in [1]. This family of Lie superalgebras provides us with a realization of an arbitrary generalized root graded Lie superalgebra of type $P(m-1)$ for $m \neq 4$ up to central isogeneous (c.f. [5]), which is a complement to the realization of a root graded Lie superalgebra of type $P(m-1)$ given in [14].

A primary result of this paper is Theorem 5.5 which states that the second homology of the Lie superalgebra $\mathfrak{p}_m(R, ^-)$ with $m \geqslant 5$ is isomorphic to the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, ^- \circ \rho)$, where $^- \circ \rho$ is the superinvolution on $R$ obtained by twisting the superinvolution $^-$ with the sign map $\rho$ (see (2.5) in Section 2). In the special case where $R$ is super-commutative, the isomorphism indicates that the second homology of $\mathfrak{p}_m(\Bbbk) \otimes_{\Bbbk} R$ for a super-commutative superalgebra $R$ is trivial, which was obtained by K. Iohara and Y. Koga in [7, 8]. While the isomorphism also reveals that the second homology of $\mathfrak{p}_m(R, ^-)$ is not necessarily trivial if $R$ is not super-commutative.

The methods used in this paper non-suprisingly involve explicitly construction of the universal central extension of $\mathfrak{p}_m(R, ^-)$, which will be achieved via introducing the notion of Steinberg periplectic Lie superalgebra $\mathfrak{stp}_m(R, ^-)$ in Section 3.

The isomorphism between the second homology of $\mathfrak{p}_m(R, ^-)$ and the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, ^- \circ \rho)$ fails when $m = 3$ or $4$. Nonetheless, the second homology of $\mathfrak{p}_4(R, ^-)$ and $\mathfrak{p}_3(R, ^-)$ will also been explicitly computed in Theorems 6.3 and 7.3.

## 2 Basics on generalized periplectic Lie superalgebras

We briefly review the definition of a generalized periplectic Lie superalgebra and prove a few properties in this section.

Throughout the paper, we always assume $\Bbbk$ is a commutative base ring with 2 invertible. All modules, associative superalgebras and Lie superalgebras are assumed to be over $\Bbbk$. Let $R$ be a unital associative superalgebra, in which the parity of $a \in R$ is denoted by $|a|$. Then the associative superalgebra $\mathrm{M}_{m|m}(R)$ of all $2m \times 2m$-matrices is also equipped with a $\mathbb{Z}/2\mathbb{Z}$-gradation by setting

$$|e_{ij}(a)| := |i| + |j| + |a|, \quad a \in R, \quad 1 \leqslant i, j \leqslant 2m, \tag{2.1}$$

where $e_{ij}(a)$ is the matrix unit with $a$ at the $(i, j)$-position and 0 elsewhere,

$$|i| = \begin{cases} 0, & \text{if } i \leqslant m, \\ 1, & \text{if } i > m. \end{cases} \tag{2.2}$$

This makes $\mathrm{M}_{m|m}(R)$ an associative superalgebra.

We assume in addition that $R$ is equipped with a superinvolution[1] $^- : R \to R$ that is a $\Bbbk$-linear map satisfying

$$\overline{ab} = (-1)^{|a||b|}\bar{b}\bar{a}, \text{ and } \bar{\bar{a}} = a, \tag{2.3}$$

for homogeneous $a, b \in R$. This further gives rise to a periplectic superinvolution on the associative superalgebra $\mathrm{M}_{m|m}(R)$ defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{prp}} := \begin{pmatrix} \overline{D}^t & -\overline{\rho(B)}^t \\ \overline{\rho(C)}^t & \overline{A}^t \end{pmatrix}, \tag{2.4}$$

where $A, B, C, D$ are $m \times m$-matrices with entries in $R$, $\rho : R \to R$ is the $\Bbbk$-linear map defined by

$$\rho(a) = (-1)^{|a|}a, \tag{2.5}$$

[1] The superinvolutions on the matrix superalgebra $\mathrm{M}_{m|n}(\Bbbk)$ over a field $\Bbbk$ of characteristic not 2 were classified in [16]. A superinvolution on $\mathrm{M}_{m|n}(\Bbbk)$ may not exist. Whenever it exists, a superinvolution on $\mathrm{M}_{m|n}(\Bbbk)$ is equivalent to either a periplectic superinvolution or an orthosymplectic superinvolution. This motivates us to define the periplectic superinvolution on $\mathrm{M}_{m|m}(R)$ here.

for homogeneous $a \in R$, $\rho(A)$ denotes the matrix $(\rho(a_{ij}))$ and $\overline{A} = (\overline{a_{ij}})$ for $A = (a_{ij})$. In this situation, one defines a Lie superalgebra

$$\widetilde{\mathfrak{p}}_m(R, ^-) := \{X \in \mathrm{M}_{m|m}(R) | X^{\mathrm{prp}} = -X\}, \tag{2.6}$$

with the standard super-commutator as the super-bracket. Its derived Lie sub-superalgebra

$$\mathfrak{p}_m(R, ^-) := [\widetilde{\mathfrak{p}}_m(R, ^-), \widetilde{\mathfrak{p}}_m(R, ^-)] \tag{2.7}$$

is called *the generalized periplectic Lie superalgebra coordinatized by the associative superalgebra* $(R, ^-)$ *with superinvolution.*

As an example, we consider $R = \Bbbk$ on which the identity map is a superinvolution. The Lie superalgebra $\mathfrak{p}_m(\Bbbk, \mathrm{id})$ coincides with the simple Lie superalgebra of type $P(m-1)$ as defined in [9]. We simply write $\mathfrak{p}_m(\Bbbk) := \mathfrak{p}_m(\Bbbk, \mathrm{id})$.

If $R$ is super-commutative, there is a natural Lie superalgebra structure on $\mathfrak{p}_m(\Bbbk) \otimes_{\Bbbk} R$ with the super-bracket

$$[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x, y] \otimes ab, \tag{2.8}$$

for homogeneous $x, y \in \mathfrak{p}_m(\Bbbk)$ and $a, b \in R$. The Lie superalgebra $\mathfrak{p}_m(\Bbbk)\otimes_{\Bbbk} R$ is actually isomorphic to the generalized periplectic Lie superalgebra $\mathfrak{p}_m(R, \rho)$, where the $\Bbbk$-linear map $\rho : R \to R$ (2.5) is a superinvolution on $R$ when $R$ is super-commutative.

Before going into the discussion on the properties of generalized periplectic Lie superalgebras, we exhibit another example here:

**Example 2.1.** *Let $S$ be an arbitrary unital associative superalgebra and $S^{\mathrm{op}}$ denote its opposite superalgebra with the multiplication*

$$a \overset{\mathrm{op}}{\cdot} b = (-1)^{|a||b|} b \cdot a, \tag{2.9}$$

*for homogeneous $a, b \in S$. Then the $\Bbbk$-linear map*

$$\mathrm{ex} : S \oplus S^{\mathrm{op}} \to S \oplus S^{\mathrm{op}}, \quad a \oplus b \mapsto b \oplus a. \tag{2.10}$$

*is a superinvolution on $S \oplus S^{\mathrm{op}}$. In this situation, we have an isomorphism of Lie superalgebras*

$$\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex}) \cong \mathfrak{sl}_{m|m}(S) := [\mathfrak{gl}_{m|m}(S), \mathfrak{gl}_{m|m}(S)], \quad m \geqslant 1, \tag{2.11}$$

*where $\mathfrak{gl}_{m|m}(S)$ is the Lie superalgebra of $2m \times 2m$-matrices with entrices in $S$.*

*Proof.* In fact, the Lie superalgebra $\widetilde{\mathfrak{p}}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ is isomorphic to the Lie superalgebra $\mathfrak{gl}_{m|m}(S)$, where an explicit isomorphism $\mathfrak{gl}_{m|m}(S) \to \tilde{\mathfrak{p}}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ is given as follows:

$$\begin{aligned}
e_{i,j}(a) &\mapsto e_{i,j}(a \oplus 0) - e_{m+j,m+i}(0 \oplus a), \\
e_{i,m+j}(a) &\mapsto e_{i,m+j}(a \oplus 0) + e_{j,m+i}(0 \oplus \rho(a)), \\
e_{m+i,j}(a) &\mapsto e_{m+i,j}(a \oplus 0) - e_{m+j,i}(0 \oplus \rho(a)), \\
e_{m+i,m+j}(a) &\mapsto -e_{j,i}(0 \oplus a) + e_{m+i,m+j}(a \oplus 0),
\end{aligned}$$

for $a \in S$ and $1 \leqslant i, j \leqslant m$. Taking their derived Lie sub-superalgebras, we conclude that the Lie superalgebra $\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ is isomorphic to the Lie superalgebra $\mathfrak{sl}_{m|m}(S)$. □

It is known from [15] that a Lie superalgebra admits a universal central extension if and only if it is perfect. In order to discuss the universal central extension of the generalized periplectic Lie superalgebra $\mathfrak{p}_m(R, ^-)$, we explore the perfectness of $\mathfrak{p}_m(R, ^-)$. We will use the following notation:

$$t_{ij}(a) := e_{ij}(a) - e_{m+j,m+i}(\bar{a}), \tag{2.12}$$

$$f_{ij}(a) := e_{i,m+j}(a) + e_{j,m+i}(\rho(\bar{a})), \tag{2.13}$$

$$g_{ij}(a) := e_{m+i,j}(a) - e_{m+j,i}(\rho(\bar{a})). \tag{2.14}$$

We always denote $R_{(\pm)} := \{a \in R | \bar{a} = \pm\rho(a)\}$.

**Lemma 2.2.** *For $m \geqslant 2$, every element of $x \in \tilde{\mathfrak{p}}_m(R, {}^-)$ is written as*

$$\begin{aligned} x = & t_{11}(a) + \sum_{i=2}^{m}(t_{ii}(a_i) - t_{11}(a_i)) + \sum_{1\leqslant i\neq j\leqslant m} t_{ij}(a_{ij}) \\ & + \sum_{i=1}^{m}(e_{i,m+i}(b_i) + e_{m+i,i}(c_i)) + \sum_{1\leqslant i<j\leqslant m} (f_{ij}(b_{ij}) + g_{ij}(c_{ij})), \end{aligned} \tag{2.15}$$

*where $a, a_i, a_{ij}, b_{ij}, c_{ij} \in R$, $b_i \in R_{(+)}$ and $c_i \in R_{(-)}$ are uniquely determined by $x$. Moreover, such an element $x$ is contained in $\mathfrak{p}_m(R, {}^-) = [\widetilde{\mathfrak{p}}_m(R, {}^-), \widetilde{\mathfrak{p}}_m(R, {}^-)]$ if and only if $a \in [R, R] + R_{(-)}$.*

*Proof.* The first statement follows from the definition of $\tilde{\mathfrak{p}}_m(R, {}^-)$. We show that $x \in \mathfrak{p}_m(R, {}^-)$ if and only if $a \in [R, R] + R_{(-)}$.

We observe that each term on the right hand side of (2.15) except $t_{11}(a)$ is a super-commutator of two elements in $\widetilde{\mathfrak{p}}_m(R, {}^-)$, i.e., they are contained in $\mathfrak{p}_m(R, {}^-) = [\widetilde{\mathfrak{p}}_m(R, {}^-), \widetilde{\mathfrak{p}}_m(R, {}^-)]$. Hence, it suffices to show that $t_{11}(a) \in \mathfrak{p}_m(R, {}^-)$ if and only if $a \in [R, R] + R_{(-)}$.

If $a \in [R, R]$, we write $a = \sum [a'_i, a''_i]$ with $a'_i, a''_i \in R$, then

$$t_{11}(a) = \sum [t_{11}(a'_i), t_{11}(a''_i)] \in \mathfrak{p}_m(R, {}^-).$$

While an element $a \in R_{(-)}$ satisfies that

$$t_{11}(a) + t_{22}(a) = t_{11}(a) - t_{22}(\rho(\bar{a})) = [f_{12}(a), g_{21}(1)] \in \mathfrak{p}_m(R, {}^-).$$

Combining with $t_{22}(a) - t_{11}(a) \in \mathfrak{p}_m(R, {}^-)$ and $\frac{1}{2} \in \Bbbk$, we conclude that $t_{11}(a) \in \mathfrak{p}_m(R, {}^-)$. This shows that $t_{11}(a) \in \mathfrak{p}_m(R, {}^-)$ if $a \in [R, R] + R_{(-)}$.

For the inverse implication, we observe that every element

$$\begin{pmatrix} A & B \\ C & -\overline{A}^t \end{pmatrix} \in \mathfrak{p}_m(R, {}^-) = [\tilde{\mathfrak{p}}_m(R, {}^-), \tilde{\mathfrak{p}}_m(R, {}^-)]$$

satisfies $\mathrm{Tr}(A) \in [R, R] + R_{(-)}$. Hence, $a \in [R, R] + R_{(-)}$ if $t_{11}(a) \in \mathfrak{p}_m(R, {}^-)$. □

**Proposition 2.3.** *Let $(R, {}^-)$ be a unital associative superalgebra with superinvolution and $m \geqslant 2$.*

*(i) There is an exact sequence of Lie superalgebras*

$$0 \to \mathfrak{p}_m(R, {}^-) \to \tilde{\mathfrak{p}}_m(R, {}^-) \to \frac{R}{[R, R] + R_{(-)}} \to 0.$$

*(ii) The Lie superalgebra $\mathfrak{p}_m(R, {}^-)$ is generated by $t_{ij}(a)$, $f_{ij}(a)$, $g_{ij}(a)$ for $a \in R$, $1 \leqslant i \neq j \leqslant m$.*

*(iii) If $m \geqslant 3$, then the Lie superalgebra $\mathfrak{p}_m(R, {}^-)$ is perfect, i.e.,*

$$\mathfrak{p}_m(R, {}^-) = [\mathfrak{p}_m(R, {}^-), \mathfrak{p}_m(R, {}^-)].$$

*Proof.* (i) We define a surjective $\Bbbk$–linear map

$$\eta : \tilde{\mathfrak{p}}_m(R, {}^-) \to \frac{R}{[R, R] + R_{(-)}}, \quad \begin{pmatrix} A & B \\ C & -\overline{A}^t \end{pmatrix} \mapsto \mathrm{Tr}(A) + ([R, R] + R_{(-)}).$$

By Lemma 2.2, $\ker \eta = \mathfrak{p}_m(R, {}^-)$. Hence, we obtain an exact sequence of $\Bbbk$–modules:

$$0 \to \mathfrak{p}_m(R, {}^-) \to \tilde{\mathfrak{p}}_m(R, {}^-) \to \frac{R}{[R, R] + R_{(-)}} \to 0.$$

Note that $R/([R, R] + R_{(-)})$ is a super-commutative Lie superalgebra, we obtain that all $\Bbbk$–linear maps appearing in this exact sequence are homomorphisms of Lie superalgebras.

(ii) By Lemma 2.2, it suffices to show $t_{11}(a)$ with $a \in [R,R]+R_{(-)}$, $t_{ii}(a)-t_{11}(a)$ with $a\in R, 2\leqslant i\leqslant m$, $e_{i,m+i}(a)$ with $a\in R_{(+)}, 1\leqslant i\leqslant m$, and $e_{m+i,i}(a)$ with $a\in R, 1\leqslant i\leqslant m$ can be generated by $t_{ij}(b)$, $f_{ij}(b)$ and $g_{ij}(b)$ for $b\in R$ and $1\leqslant i\neq j\leqslant m$. Indeed,

$$t_{ii}(a)-t_{11}(a)=[t_{i1}(a),t_{1i}(1)],\quad e_{i,m+i}(a)=\frac{1}{2}[t_{ij}(1),f_{ji}(a)],\text{ and } e_{m+i,i}(a)=\frac{1}{2}[g_{ij}(a),t_{ji}(1)],$$

where $1\leqslant j\leqslant m$ is chosen such that $i\neq j$. Furthermore, for $a\in[R,R]+R_{(-)}$, we also have

$$\begin{aligned} t_{11}([a',a''])&=[t_{12}(a'),t_{21}(a'')]-(-1)^{|a'||a''|}[t_{12}(1),t_{21}(a''a')], && a',a''\in R\\ t_{11}(a)&=\frac{1}{2}[t_{12}(1),t_{21}(a)]+\frac{1}{2}[f_{12}(1),g_{21}(a)], && a\in R_{(-)}.\end{aligned}$$

This proves (ii).

(iii) By (ii), $\mathfrak{p}_m(R,^-)$ is generated by $t_{ij}(a), f_{ij}(a), g_{ij}(a)$ with $a\in R$ and $1\leqslant i\neq j\leqslant m$. We shall show that these elements are also contained in $[\mathfrak{p}_m(R,^-),\mathfrak{p}_m(R,^-)]$. Note that $m\geqslant 3$, for $1\leqslant i\neq j\leqslant m$, we may choose $1\leqslant k\leqslant m$ such that $i,j,k$ are distinct. Then the equalities

$$t_{ij}(a)=[t_{ik}(1),t_{kj}(a)],\quad f_{ij}(a)=[t_{ik}(1),f_{kj}(a)],\quad g_{ij}(a)=[g_{ik}(a),t_{kj}(1)]$$

imply that $t_{ij}(a), f_{ij}(a), g_{ij}(a)\in[\mathfrak{p}_m(R,^-),\mathfrak{p}_m(R,^-)]$. Hence, $\mathfrak{p}_m(R,^-)$ is perfect for $m\geqslant 3$. $\square$

*Remark* 2.4. The Lie superalgebra $\mathfrak{p}_1(R,^-)$ is not necessarily perfect. For instance, if $R$ is supercommutative, then

$$\tilde{\mathfrak{p}}_1(R,\rho):=\left\{\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}\middle| a,b,c\in R\right\},\text{ and } \mathfrak{p}_1(R,\rho)=\left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\middle| a,b,c\in R\right\}.$$

The Lie superalgebra $\mathfrak{p}_1(R,\rho)$ is not perfect since $[\mathfrak{p}_1(R,\rho),\mathfrak{p}_1(R,\rho)]=0$. In general, the condition for the perfectness of $\mathfrak{p}_1(R,^-)$ is unknown yet.

Similarly, the Lie superalgebra $\mathfrak{p}_2(R,^-)$ is also not necessarily perfect. Hence, the existence of a universal central extension of $\mathfrak{p}_1(R,^-)$ or $\mathfrak{p}_2(R,^-)$ is not ensured. We only consider the second homology of $\mathfrak{p}_m(R,^-)$ for $m\geqslant 3$.

# 3 Steinberg periplectic Lie superalgebras

In the previous section, we have shown that the Lie superalgebra $\mathfrak{p}_m(R,^-)$ is perfect for $m\geqslant 3$. The perfectness allows us to further study its universal central extension, whose kernel will finally provides us with the second homology of $\mathfrak{p}_m(R,^-)$.

In this section, we will introduce the notion of Steinberg periplectic Lie superalgebra $\mathfrak{stp}_m(R,^-)$ and prove that it is a central extension of $\mathfrak{p}_m(R,^-)$. Its universality will be dicussed in the sequent sections.

**Definition 3.1.** Let $(R,^-)$ be a unital associative superalgebra with superinvolution and $m\geqslant 3$. *The Steinberg periplectic Lie superalgebra coordinatized by* $(R,^-)$, denoted by $\mathfrak{stp}_m(R,^-)$, is defined to be the abstract Lie superalgebra generated by homogenous elements $\mathbf{t}_{ij}(a)$, $\mathbf{f}_{ij}(a)$, $\mathbf{g}_{ij}(a)$ with parity $|a|$, $|a|+1$, $|a|+1$ respectively, for homogeneous $a\in R$ and $1\leqslant i\neq j\leqslant m$, subjecting to the relations:

$$\begin{aligned}
&\mathbf{t}_{ij},\mathbf{f}_{ij},\mathbf{g}_{ij}\text{ are all }\Bbbk\text{-linear}, && \text{for } i\neq j, && \text{(STP00)}\\
&\mathbf{f}_{ij}(\bar a)=\mathbf{f}_{ji}(\rho(a)), && \text{for } i\neq j, && \text{(STP01)}\\
&\mathbf{g}_{ij}(\bar a)=-\mathbf{g}_{ji}(\rho(a)), && \text{for } i\neq j, && \text{(STP02)}\\
&[\mathbf{t}_{ij}(a),\mathbf{t}_{jk}(b)]=\mathbf{t}_{ik}(ab), && \text{for distinct } i,j,k, && \text{(STP03)}\\
&[\mathbf{t}_{ij}(a),\mathbf{t}_{kl}(b)]=0, && \text{for } i\neq j\neq k\neq l\neq i, && \text{(STP04)}
\end{aligned}$$

$$
\begin{aligned}
&[\mathbf{t}_{ij}(a), \mathbf{f}_{jk}(b)] = \mathbf{f}_{ik}(ab), && \text{for distinct } i, j, k, && \text{(STP05)}\\
&[\mathbf{t}_{ij}(a), \mathbf{f}_{kl}(b)] = 0, && \text{for } i \neq j \neq k \neq l \neq j, && \text{(STP06)}\\
&[\mathbf{g}_{ij}(a), \mathbf{t}_{jk}(b)] = \mathbf{g}_{ik}(ab), && \text{for distinct } i, j, k, && \text{(STP07)}\\
&[\mathbf{g}_{ij}(a), \mathbf{t}_{kl}(b)] = 0, && \text{for } l \neq k \neq j \neq i \neq k, && \text{(STP08)}\\
&[\mathbf{f}_{ij}(a), \mathbf{f}_{kl}(b)] = 0, && \text{for } i \neq j, \text{ and } k \neq l, && \text{(STP09)}\\
&[\mathbf{g}_{ij}(a), \mathbf{g}_{kl}(b)] = 0, && \text{for } i \neq j, \text{ and } k \neq l, && \text{(STP10)}\\
&[\mathbf{f}_{ij}(a), \mathbf{g}_{jk}(b)] = \mathbf{t}_{ik}(ab), && \text{for distinct } i, j, k, && \text{(STP11)}\\
&[\mathbf{f}_{ij}(a), \mathbf{g}_{kl}(b)] = 0, && \text{for distinct } i, j, k, l, && \text{(STP12)}
\end{aligned}
$$

where $a, b \in R$ and $1 \leqslant i, j, k, l \leqslant m$.

Recall Proposition 2.3 that $\mathfrak{p}_m(R, ^-)$ is generated by $t_{ij}(a), f_{ij}(a)$ and $g_{ij}(a)$ for $a \in R$ and $1 \leqslant i \neq j \leqslant m$. These generators satisfy all relations (STP00)-(STP12). Hence, there is a canonical homomorphism of Lie superalgebras:

$$\psi : \mathfrak{stp}_m(R, ^-) \to \mathfrak{p}_m(R, ^-), \tag{3.1}$$

such that $\psi(\mathbf{t}_{ij}(a)) = t_{ij}(a), \psi(\mathbf{f}_{ij}(a)) = f_{ij}(a)$ and $\psi(\mathbf{g}_{ij}(a)) = g_{ij}(a)$, which will be demonstrated to be a central extension, i.e., the kernel of $\psi$ is included in the center of $\mathfrak{stp}_m(R, ^-)$.

It is easy to observe that all diagonal, upper triangular and lower triangular matrices in $\mathfrak{p}_m(R, ^-)$ form three Lie sub-superalgebras of $\mathfrak{p}_m(R, ^-)$, respectively. Their direct sum gives a decomposition of $\mathfrak{p}_m(R, ^-)$. We first show that the Steinberg periplectic Lie superalgebra $\mathfrak{stp}_m(R, ^-)$ also possesses a similar decomposition.

**Lemma 3.2.** *In the Lie superalgebra $\mathfrak{stp}_m(R, ^-)$, the following equalities hold:*

$$[\mathbf{t}_{ij}(a), \mathbf{f}_{ji}(b)] = [\mathbf{t}_{ik}(a), \mathbf{f}_{ki}(b)], \ \textit{and}\ [\mathbf{g}_{ij}(a), \mathbf{t}_{ji}(b)] = [\mathbf{g}_{ik}(a), \mathbf{t}_{ki}(b)],$$

*for $a, b \in R$ and $1 \leqslant i, j, k \leqslant m$ with $i \neq j, k$.*

*Proof.* We assume $i, j, k$ are distinct and deduce from (STP03), (STP05) and (STP06) that

$$
\begin{aligned}
[\mathbf{t}_{ik}(a), \mathbf{f}_{ki}(b)] &= [[\mathbf{t}_{ij}(a), \mathbf{t}_{jk}(1)], \mathbf{f}_{ki}(b)]\\
&= [[\mathbf{t}_{ij}(a), \mathbf{f}_{ki}(b)], \mathbf{t}_{jk}(1)] + [\mathbf{t}_{ij}(a), [\mathbf{t}_{jk}(1), \mathbf{f}_{ki}(b)]]\\
&= 0 + [\mathbf{t}_{ij}(a), \mathbf{f}_{ji}(b)]\\
&= [\mathbf{t}_{ij}(a), \mathbf{f}_{ji}(b)].
\end{aligned}
$$

Similarly, $[\mathbf{g}_{ij}(a), \mathbf{t}_{ji}(b)] = [\mathbf{g}_{ik}(a), \mathbf{t}_{ki}(b)]$ follows from (STP03), (STP07) and (STP08). □

Lemma 3.2 permits us to introduce the following well-defined elements of $\mathfrak{stp}_m(R, ^-)$:

$$
\begin{aligned}
\mathbf{f}_i(a) &:= [\mathbf{t}_{ij}(1), \mathbf{f}_{ji}(a)], && \text{for some } j \neq i, && (3.2)\\
\mathbf{g}_i(a) &:= [\mathbf{g}_{ij}(a), \mathbf{t}_{ji}(1)], && \text{for some } j \neq i, && (3.3)\\
\mathbf{h}_{ij}(a, b) &:= [\mathbf{f}_{ij}(a), \mathbf{g}_{ji}(b)], && \text{for } i \neq j, && (3.4)
\end{aligned}
$$

where $a, b \in R$ and $1 \leqslant i, j \leqslant m$. One easily deduce that

$$\boldsymbol{f}_i(\bar{a}) = \boldsymbol{f}_i(\rho(a)), \text{ and } \boldsymbol{g}_i(\bar{a}) = -\boldsymbol{g}_i(\rho(a)), \tag{3.5}$$

for $1 \leqslant i \leqslant m$ and $a \in R$.

**Proposition 3.3.** *The Lie superalgebra $\mathfrak{stp}_m(R, ^-)$ is decomposed as a direct sum of $\Bbbk$-modules:*

$$\mathfrak{stp}_m(R, ^-) = \mathfrak{stp}_m^-(R, ^-) \oplus \mathfrak{stp}_m^0(R, ^-) \oplus \mathfrak{stp}_m^+(R, ^-), \tag{3.6}$$

*where*

$$\begin{aligned}
\mathfrak{stp}^0_m(R,^-) &:= \mathrm{span}_{\Bbbk}\{\mathbf{h}_{ij}(a,b)|a,b\in R, 1\leqslant i\neq j\leqslant m\},\\
\mathfrak{stp}^+_m(R,^-) &:= \mathrm{span}_{\Bbbk}\{\mathbf{t}_{ij}(a),\mathbf{f}_{ij}(a),\mathbf{f}_k(a)|a\in R, 1\leqslant i,j,k\leqslant m \textit{ and } i<j\},\\
\mathfrak{stp}^-_m(R,^-) &:= \mathrm{span}_{\Bbbk}\{\mathbf{t}_{ij}(a),\mathbf{g}_{ij}(a),\mathbf{g}_k(a)|a\in R, 1\leqslant i,j,k\leqslant m \textit{ and } i>j\},
\end{aligned}$$

*are all Lie sub-superalgebras of* $\mathfrak{stp}_m(R,^-)$ *and* $[\mathfrak{stp}^0_m(R,^-),\mathfrak{stp}^\pm_m(R,^-)]\subseteq\mathfrak{stp}^\pm_m(R,^-)$.

*Proof.* We first deduce from (STP00)-(STP12) that $\mathfrak{stp}^0_m(R,^-)$, $\mathfrak{stp}^-_m(R,^-)$ and $\mathfrak{stp}^+_m(R,^-)$ are all Lie sub-superalgebras of $\mathfrak{stp}_m(R,^-)$ and

$$[\mathfrak{stp}^0_m(R,^-),\mathfrak{stp}^\pm_m(R,^-)]\subseteq\mathfrak{stp}^\pm_m(R,^-).$$

Next, we denote $\mathfrak{g}:=\mathfrak{stp}^-_m(R,^-)+\mathfrak{stp}^0_m(R,^-)+\mathfrak{stp}^+_m(R,^-)$ and show that $\mathfrak{stp}_m(R,^-)=\mathfrak{g}$. The $\Bbbk$–module $\mathfrak{g}$ is invariant under $\mathrm{ad}(\mathbf{t}_{ij}(a))$, $\mathrm{ad}(\mathbf{f}_{ij}(a))$, and $\mathrm{ad}(\mathbf{g}_{ij}(a))$. Note that $\mathbf{t}_{ij}(a)$, $\mathbf{f}_{ij}(a)$, and $\mathbf{g}_{ij}(a)$ with $a\in R$ and $1\leqslant i\neq j\leqslant m$ generate the Lie superalgebra $\mathfrak{stp}_m(R,^-)$, we obtain that $\mathfrak{g}$ is an ideal of the Lie superalgebra $\mathfrak{stp}_m(R,^-)$. It follows that $\mathfrak{stp}_m(R,^-)=\mathfrak{g}$ since $\mathfrak{g}$ contains a complete family of generators of $\mathfrak{stp}_m(R,^-)$.

Finally, we prove that the summation in the decomposition (3.6) is a direct sum. We claim that the restriction $\psi|\mathfrak{stp}^\pm_m(R,^-)$ of the canonical homomorphism (3.1) is injective. Suppose that $\boldsymbol{x}^+\in\mathfrak{stp}^+_m(R,^-)$ satisfying $\psi(\boldsymbol{x}^+)=0$. Write

$$\boldsymbol{x}^+=\sum_{1\leqslant i<j\leqslant m}\mathbf{t}_{ij}(a_{ij})+\sum_{1\leqslant i<j\leqslant m}\mathbf{f}_{ij}(b_{ij})+\sum_i\mathbf{f}_i(c_i),$$

where $a_{ij},b_{ij},c_i\in R$. Applying $\psi$, we obtain

$$0=\psi(\boldsymbol{x}^+)=\sum_{1\leqslant i<j\leqslant m}(t_{ij}(a_{ij})+f_{ij}(b_{ij}))+\sum_{i=1}^m e_{i,m+i}(c_i+\rho(\bar{c}_i))\in\mathfrak{p}_m(R,^-).$$

It follows that $a_{ij}=b_{ij}=0$ for $1\leqslant i<j\leqslant m$ and $c_i+\rho(\bar{c}_i)=0$ for $i=1,\ldots,m$. Now,

$$\begin{aligned}
\boldsymbol{x}^+&=\sum_{1\leqslant i<j\leqslant m}\mathbf{t}_{ij}(a_{ij})+\sum_{1\leqslant i<j\leqslant m}\mathbf{f}_{ij}(b_{ij})+\sum_i\mathbf{f}_i(c_i)\\
&=\sum_{1\leqslant i<j\leqslant m}\mathbf{t}_{ij}(a_{ij})+\sum_{1\leqslant i<j\leqslant m}\mathbf{f}_{ij}(b_{ij})+\sum_i\frac{1}{2}(\mathbf{f}_i(c_i)+\mathbf{f}_i(\rho(c_i)))=0,
\end{aligned}$$

since $\boldsymbol{f}_i(\bar{c})=\boldsymbol{f}_i(\rho(c))$. Hence, $\psi|\mathfrak{stp}^+_m(R,^-)$ is injective. Similarly, $\psi|\mathfrak{stp}^-_m(R,^-)$ is injective.

Now, if $0=\boldsymbol{x}^-+\boldsymbol{x}^0+\boldsymbol{x}^+$ for $\boldsymbol{x}^0\in\mathfrak{stp}^0_m(R,^-)$ and $\boldsymbol{x}^\pm\in\mathfrak{stp}^\pm_m(R,^-)$, then

$$0=\psi(\boldsymbol{x}^-)+\psi(\boldsymbol{x}^0)+\psi(\boldsymbol{x}^+)$$

in $\mathfrak{p}_m(R,^-)$, where $\psi(\boldsymbol{x}^-)$ (resp. $\psi(\boldsymbol{x}^0)$ or $\psi(\boldsymbol{x}^+)$) is a lower-triangular (resp. diagonal or upper-triangular) matrix. It follows that $\psi(\boldsymbol{x}^-)=\psi(\boldsymbol{x}^0)=\psi(\boldsymbol{x}^+)=0$ and yields $\boldsymbol{x}^-=\boldsymbol{x}^+=0$ since $\psi|\mathfrak{stp}^\pm_m(R,^-)$ is injective. Hence, the summation (3.6) is a direct sum. □

**Proposition 3.4.** *Let* $(R,^-)$ *be a unital associative superalgebra with superinvolution and* $m\geqslant 3$. *Then* $\psi:\mathfrak{stp}_m(R,^-)\to\mathfrak{p}_m(R,^-)$ *is a central extension and* $\ker\psi\subseteq\mathfrak{stp}^0_m(R,^-)$.

*Proof.* Let $\boldsymbol{x}\in\ker\psi$. We write $\boldsymbol{x}=\boldsymbol{x}^-+\boldsymbol{x}^0+\boldsymbol{x}^+$ with respect to the decomposition (3.6). Then

$$0=\psi(\boldsymbol{x})=\psi(\boldsymbol{x}^-)+\psi(\boldsymbol{x}^0)+\psi(\boldsymbol{x}^+)\in\mathfrak{p}_m(R,^-),$$

where $\psi(\boldsymbol{x}^-)$ (resp. $\psi(\boldsymbol{x}^0)$ or $\psi(\boldsymbol{x}^+)$) is a lower-triangular (resp. diagonal or upper-triangular) matrix. Hence, $\psi(\boldsymbol{x}^-)=\psi(\boldsymbol{x}^0)=\psi(\boldsymbol{x}^+)=0$. Recall from the proof of Proposition 3.3 that $\psi|\mathfrak{stp}^\pm_m(R,^-)$ are injective. It follows that $\boldsymbol{x}^+=\boldsymbol{x}^-=0$. Hence, $\boldsymbol{x}=\boldsymbol{x}^0\in\mathfrak{stp}^0_m(R,^-)$.

It remains to show that every element $\boldsymbol{x} \in \ker \psi$ commutes with the generators $\mathbf{t}_{ij}(a)$, $\mathbf{f}_{ij}(a)$ and $\mathbf{g}_{ij}(a)$. For $\boldsymbol{x} \in \ker \psi$, we have

$$\psi([\boldsymbol{x}, \mathbf{t}_{ij}(a)]) = \psi([\boldsymbol{x}, \mathbf{f}_{ij}(a)]) = \psi([\boldsymbol{x}, \mathbf{g}_{ij}(a)]) = 0.$$

Note that $\boldsymbol{x} \in \ker \psi \subseteq \mathfrak{stp}_m^0(R, ^-)$, it follows from Proposition 3.3 that $[\boldsymbol{x}, \mathbf{t}_{ij}(a)]$, $[\boldsymbol{x}, \mathbf{f}_{ij}(a)]$ and $[\boldsymbol{x}, \mathbf{g}_{ij}(a)]$ are all contained in either $\mathfrak{stp}_m^+(R, ^-)$ or $\mathfrak{stp}_m^-(R, ^-)$. Hence,

$$[\boldsymbol{x}, \mathbf{t}_{ij}(a)] = [\boldsymbol{x}, \mathbf{f}_{ij}(a)] = [\boldsymbol{x}, \mathbf{g}_{ij}(a)] = 0$$

since $\psi|\mathfrak{stp}_m^{\pm}(R, ^-)$ are injective. □

In the special case where $(R, ^-) = (S \oplus S^{\mathrm{op}}, \mathrm{ex})$ for a unital associative superalgebra $S$, Example 2.1 implies that the Lie superalgebra $\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ is isomorphic to $\mathfrak{sl}_{m|m}(S)$. According to Proposition 3.4, a central extension of $\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ is given by $\mathfrak{stp}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$, which is isomorphic to *the Steinberg Lie superalgebra* $\mathfrak{st}_{m|m}(S)$ defined in [4]:

**Proposition 3.5.** *Let $S$ be an arbitrary unital associative superalgebra and $m \geqslant 3$. Then*

$$\mathfrak{stp}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex}) \cong \mathfrak{st}_{m|m}(S)$$

*as Lie superalgebras over $\Bbbk$.*

*Proof.* The Steinberg Lie superalgebra $\mathfrak{st}_{m|m}(S)$ as defined in [4] is the abstract Lie superalgebra generated by homogeneous elements $\boldsymbol{e}_{ij}(a)$ of degree $|i|+|j|+|a|$ for $a \in R$ and $1 \leqslant i \neq j \leqslant m+n$, subjecting to the relations:

$$\begin{aligned}
&a \mapsto \boldsymbol{e}_{ij}(a) \text{ is } \Bbbk\text{-linear}, && && \text{(ST0)}\\
&[\boldsymbol{e}_{ij}(a), \boldsymbol{e}_{jk}(b)] = \boldsymbol{e}_{ik}(ab), && \text{for distinct } i, j, k, && \text{(ST1)}\\
&[\boldsymbol{e}_{ij}(a), \boldsymbol{e}_{kl}(b)] = 0, && \text{for } i \neq j \neq k \neq l \neq i, && \text{(ST2)}
\end{aligned}$$

where $a, b \in R$ and $1 \leqslant i, j, k, l \leqslant m+n$.

According to the relations (STP00)-(STP12), there is a homomorphism of Lie superalgebras $\phi : \mathfrak{st}_{m|m}(S) \to \mathfrak{stp}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$ such that

$$\begin{aligned}
\phi(\boldsymbol{e}_{ij}(a)) &:= \mathbf{t}_{ij}(a \oplus 0), & \phi(\boldsymbol{e}_{i,m+j}(a)) &:= \mathbf{f}_{ij}(a \oplus 0),\\
\phi(\boldsymbol{e}_{m+i,j}(a)) &:= \mathbf{g}_{ij}(a \oplus 0), & \phi(\boldsymbol{e}_{m+i,m+j}(a)) &:= -\mathbf{t}_{ji}(0 \oplus a),\\
\phi(\boldsymbol{e}_{i,m+i}(a)) &:= [\mathbf{t}_{ij}(1 \oplus 0), \mathbf{f}_{ji}(a \oplus 0)], & \phi(\boldsymbol{e}_{m+i,i}(a)) &:= [\mathbf{g}_{ij}(a \oplus 0), \mathbf{t}_{ji}(1 \oplus 0)],
\end{aligned}$$

for $a \in S$ and $1 \leqslant i \neq j \leqslant m$. It has an inverse $\tilde{\phi} : \mathfrak{stp}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex}) \to \mathfrak{st}_{m|m}(S)$ given by

$$\begin{aligned}
\tilde{\phi}(\boldsymbol{t}_{ij}(a \oplus b)) &= \boldsymbol{e}_{ij}(a) - \boldsymbol{e}_{m+j,m+i}(b),\\
\tilde{\phi}(\boldsymbol{f}_{ij}(a \oplus b)) &= \boldsymbol{e}_{i,m+j}(a) + \boldsymbol{e}_{j,m+i}(\rho(b)),\\
\tilde{\phi}(\boldsymbol{g}_{ij}(a \oplus b) &= \boldsymbol{e}_{m+i,j}(a) - \boldsymbol{e}_{m+j,i}(\rho(b)),
\end{aligned}$$

for $a, b \in S$ and $1 \leqslant i \neq j \leqslant m$. Hence, we obtain the desired isomorphism. □

# 4 Characterization of the kernel

We have shown that the canonical epimorphism $\psi : \mathfrak{stp}_m(R, ^-) \to \mathfrak{p}_m(R, ^-)$ is a central extension. This section is devoted to explicitly characterizing the kernel of $\psi$. Here, we need the notion of the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology ${}_{+}\mathrm{HD}_1(R, ^-)$ for a unital associative superalgebra $(R, ^-)$ with superinvolution.

The $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, ^-)$ is a natrual $\mathbb{Z}/2\mathbb{Z}$-graded analogue of the dihedral homology of a unital associative algebra with anti-involution. It can be defined through the coinvariant complex of the Hoschild complex under certain action of the dihedral group as in [11]. For the use in this paper, we only describe its degree one term here:

Let $I$ be the $\Bbbk$-submodule of $R\otimes_{\Bbbk} R$ spanned by

$$a\otimes b+(-1)^{|a||b|}b\otimes a,\quad a\otimes b+\bar{a}\otimes\bar{b},\ \text{and } (-1)^{|a||c|}ab\otimes c+(-1)^{|b||a|}bc\otimes a+(-1)^{|c||b|}ca\otimes b,$$

for homogenous $a, b, c \in R$. Let $\langle R, R\rangle := (R\otimes_{\Bbbk} R)/I$ and $\langle a, b\rangle = a\otimes b + I$, then *the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of* $(R, ^-)$ is

$$_{+}\mathrm{HD}_1(R, ^-) := \left\{ \sum_i \langle a_i, b_i\rangle \,\middle|\, \sum_i \overline{[a_i, b_i]} = -\sum_i [a_i, b_i] \right\}. \tag{4.1}$$

**Proposition 4.1.** *Let $(R, ^-)$ be a unital associative superalgebra with superinvolution, $m \geqslant 3$, and $\psi : \mathfrak{stp}_m(R, ^-) \to \mathfrak{p}_m(R, ^-)$ the canonical epimorphism (3.1). Then*

$$\ker\psi \cong {}_{+}\mathrm{HD}_1(R, ^-\circ\rho) \tag{4.2}$$

*as $\Bbbk$-modules, where $\rho$ is the $\Bbbk$-linear map (2.5) and $^-\circ\rho$ is also a superinvolution on $R$.*

In order to prove this proposition, we need a few lemmas:

**Lemma 4.2.** *The elements $\mathbf{h}_{ij}(a, b) = [\mathbf{f}_{ij}(a), \mathbf{g}_{ji}(b)] \in \mathfrak{stp}_m(R, ^-)$ satisfy*

$$\mathbf{h}_{1i}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1i}(1, ba) = \mathbf{h}_{1k}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1k}(1, ba), \tag{4.3}$$

$$\mathbf{h}_{i1}(1, a) + \mathbf{h}_{1j}(1, a) - \mathbf{h}_{ij}(1, a) = \mathbf{h}_{k1}(1, a) + \mathbf{h}_{1l}(1, a) - \mathbf{h}_{kl}(1, a). \tag{4.4}$$

*for homogenous $a, b \in R$ and $2 \leqslant i, j, k, l \leqslant m$ with $i \neq j$ and $k \neq l$.*

*Proof.* Observing that the equality (4.3) is trivial when $i = k$, we assume that $2 \leqslant i \neq k \leqslant m$.

$$\begin{aligned}
&\mathbf{h}_{1k}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1k}(1, ba)\\
&= [\mathbf{f}_{1k}(a), \mathbf{g}_{k1}(b)] - (-1)^{|a||b|}[\mathbf{f}_{1k}(1), \mathbf{g}_{k1}(ba)]\\
&= -[[\mathbf{f}_{1i}(a), \mathbf{t}_{ki}(1)], \mathbf{g}_{k1}(b)] + (-1)^{|a||b|}[[\mathbf{f}_{1i}(1), \mathbf{t}_{ki}(1)], \mathbf{g}_{k1}(ba)]\\
&= -[[\mathbf{f}_{1i}(a), \mathbf{g}_{k1}(b)], \mathbf{t}_{ki}(1)] - [\mathbf{f}_{1i}(a), [\mathbf{t}_{ki}(1), \mathbf{g}_{k1}(b)]]\\
&\quad + (-1)^{|a||b|}[[\mathbf{f}_{1i}(1), \mathbf{g}_{k1}(ba)], \mathbf{t}_{ki}(1)] + (-1)^{|a||b|}[\mathbf{f}_{1i}(1), [\mathbf{t}_{ki}(1), \mathbf{g}_{k1}(ba)]]\\
&= (-1)^{|a|+|b|}[\mathbf{t}_{ik}(\bar{a}\bar{b}), \mathbf{t}_{ki}(1)] + [\mathbf{f}_{1i}(a), \mathbf{g}_{i1}(b)]\\
&\quad - (-1)^{|a|+|b|}[\mathbf{t}_{ik}(\bar{a}\bar{b}), \mathbf{t}_{ki}(1)] - (-1)^{|a||b|}[\mathbf{f}_{1i}(1), \mathbf{g}_{i1}(ba)]\\
&= [\mathbf{f}_{1i}(a), \mathbf{g}_{i1}(b)] - (-1)^{|a||b|}[\mathbf{f}_{1i}(1), \mathbf{g}_{i1}(ba)]\\
&= \mathbf{h}_{1i}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1i}(1, ba).
\end{aligned}$$

It yields the equality (4.3).

For the equality (4.4), we first consider the case where $2 \leqslant i \neq k \leqslant m$. We have

$$[\mathbf{t}_{1i}(1), \mathbf{t}_{i1}(a)] = [[\mathbf{t}_{1k}(1), \mathbf{t}_{ki}(1)], \mathbf{t}_{i1}(a)] = [\mathbf{t}_{ki}(1), \mathbf{t}_{ik}(a)] + [\mathbf{t}_{1k}(1), \mathbf{t}_{k1}(a)].$$

On the other hand, it follows from $\boldsymbol{t}_{ij}(a) = [\boldsymbol{f}_{ik}(a), \boldsymbol{g}_{kj}(1)]$ for distinct $i, j, k$ that

$$[\boldsymbol{t}_{ij}(a), \boldsymbol{t}_{ji}(b)] = \boldsymbol{h}_{ik}(a, b) - (-1)^{|a||b|}\boldsymbol{h}_{ik}(1, ba). \tag{4.5}$$

Hence,

$$\mathbf{h}_{1j}(1, a) - \mathbf{h}_{ij}(1, a) = \mathbf{h}_{k1}(1, a) - \mathbf{h}_{i1}(1, a) + \mathbf{h}_{1l}(1, a) - \mathbf{h}_{kl}(1, a),$$

i.e., (4.4) holds when $2 \leqslant i \neq k \leqslant m$.

For $i = k$, the equality (4.4) is reduced to

$$\mathbf{h}_{1j}(1, a) - \mathbf{h}_{ij}(1, a) = \mathbf{h}_{1l}(1, a) - \mathbf{h}_{il}(1, a),$$

for distinct $i, j, l \in \{2, \ldots, m\}$, whose both sides are equal to $[\mathbf{t}_{1i}(1), \mathbf{t}_{i1}(a)]$ by (4.5). □

Lemma 4.2 ensures that

$$\boldsymbol{\lambda}(a,b) := \mathbf{h}_{1i}(a,b) - (-1)^{|a||b|}\mathbf{h}_{1i}(1,ba) \in \mathfrak{stp}_m(R,{}^-) \tag{4.6}$$

is independent of $2 \leqslant i \leqslant m$, and

$$\boldsymbol{\mu}(a) := \mathbf{h}_{i1}(1,a) + \mathbf{h}_{1j}(1,a) - \mathbf{h}_{ij}(1,a) \in \mathfrak{stp}_m(R,{}^-) \tag{4.7}$$

is independent of $2 \leqslant i \neq j \leqslant m$. Moreover, they satisfy the following properties:

**Lemma 4.3.** *For homogeneous $a,b,c \in R$, we have*

*(i)* $(-1)^{|a||c|}\boldsymbol{\lambda}(ab,c) + (-1)^{|b||a|}\boldsymbol{\lambda}(bc,a) + (-1)^{|c||b|}\boldsymbol{\lambda}(ca,b) = 0,$

*(ii)* $\boldsymbol{\lambda}(a,1) = \boldsymbol{\lambda}(1,b) = 0,$

*(iii)* $\boldsymbol{\lambda}(a,b) = -(-1)^{|a||b|}\boldsymbol{\lambda}(b,a),$

*(iv)* $\boldsymbol{\mu}(\bar{a}) = -\boldsymbol{\mu}(\rho(a)).$

*Proof.* We claim that

$$\boldsymbol{\lambda}(a,b) = [\mathbf{t}_{1j}(a),\mathbf{t}_{j1}(b)] - (-1)^{|a||b|}[\mathbf{t}_{1j}(1),\mathbf{t}_{j1}(ba)], \tag{4.8}$$

for $a,b \in R$ and $j \neq 1$. Indeed, we deduce from (4.5) that

$$\begin{aligned} [\mathbf{t}_{1j}(a),\mathbf{t}_{j1}(b)] &= \mathbf{h}_{1k}(a,b) - (-1)^{|a||b|}\mathbf{h}_{jk}(1,ba), \\ [\mathbf{t}_{1j}(1),\mathbf{t}_{j1}(ba)] &= \mathbf{h}_{1k}(1,ba) - \mathbf{h}_{jk}(1,ba), \end{aligned}$$

for some $k \neq 1, j$. Hence, (4.8) holds.

Secondly, (STP03) and the Jacobi identity yield that

$$(-1)^{|a||c|}[\mathbf{t}_{ij}(ab),\mathbf{t}_{ji}(c)] + (-1)^{|a||b|}[\mathbf{t}_{ki}(bc),\mathbf{t}_{ik}(a)] + (-1)^{|b||c|}[\mathbf{t}_{jk}(ca),\mathbf{t}_{kj}(b)] = 0, \tag{4.9}$$

for distinct $i,j,k$.

(i) We deduce from (4.8) and (4.9) that

$$\begin{aligned} &(-1)^{|c||a|}\boldsymbol{\lambda}(ab,c) \\ &= (-1)^{|c||a|}[\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] - (-1)^{|b||c|}[\mathbf{t}_{1j}(1),\mathbf{t}_{j1}(cab)] \\ &= (-1)^{|c||a|}[\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] - (-1)^{|b||c|}[\mathbf{t}_{1j}(1),[\mathbf{t}_{ji}(ca),\mathbf{t}_{i1}(b)]] \\ &= (-1)^{|c||a|}[\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] + (-1)^{|b||c|}[\mathbf{t}_{ji}(ca),\mathbf{t}_{ij}(b)] - (-1)^{|b||c|}[\mathbf{t}_{1i}(ca),\mathbf{t}_{i1}(b)] \\ &= -(-1)^{|a||b|}[\mathbf{t}_{i1}(bc),\mathbf{t}_{1i}(a)] - (-1)^{|b||c|}[\mathbf{t}_{1i}(ca),\mathbf{t}_{i1}(b)]. \end{aligned}$$

On the other hand, we also compute that

$$\begin{aligned} \boldsymbol{\lambda}(ab,c) &= [\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] - (-1)^{(|a|+|b|)|c|}[\mathbf{t}_{1j}(1),\mathbf{t}_{j1}(cab)] \\ &= [\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] - (-1)^{(|a|+|b|)|c|}[\mathbf{t}_{1j}(1),[\mathbf{t}_{ji}(c),\mathbf{t}_{i1}(ab)]] \\ &= [\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] + [\mathbf{t}_{i1}(ab),\mathbf{t}_{1i}(c)] - [\mathbf{t}_{ij}(ab),\mathbf{t}_{ji}(c)]. \end{aligned}$$

It follows from (4.9) again that

$$\begin{aligned} &(-1)^{|a||c|}\boldsymbol{\lambda}(ab,c) + (-1)^{|a||b|}\boldsymbol{\lambda}(bc,a) \\ &= -(-1)^{|a||b|}[\mathbf{t}_{i1}(bc),\mathbf{t}_{1i}(a)] - (-1)^{|b||c|}[\mathbf{t}_{1i}(ca),\mathbf{t}_{i1}(b)] \\ &\quad - (-1)^{|b||c|}[\mathbf{t}_{j1}(ca),\mathbf{t}_{1j}(b)] - (-1)^{|c||a|}[\mathbf{t}_{1j}(ab),\mathbf{t}_{j1}(c)] \\ &= (-1)^{|b||c|}([\mathbf{t}_{ji}(ca),\mathbf{t}_{ij}(b)] - [\mathbf{t}_{1i}(ca),\mathbf{t}_{i1}(b)] - [\mathbf{t}_{j1}(ca),\mathbf{t}_{1j}(b)]) \end{aligned}$$

$$= -(-1)^{|b||c|}\boldsymbol{\lambda}(ca, b).$$

This proves (i).

(ii) $\boldsymbol{\lambda}(1, b) = 0$ is obvious. Taking $b = c = 1$ in (i), we obtain

$$\boldsymbol{\lambda}(a, 1) + \boldsymbol{\lambda}(1, a) + \boldsymbol{\lambda}(a, 1) = 0$$

which implies $\boldsymbol{\lambda}(a, 1) = 0$ since $\frac{1}{2} \in \Bbbk$ and $\boldsymbol{\lambda}(1, a) = 0$.

(iii) follows from (i) by taking $c = 1$.

(iv) follows from the equality $\mathbf{h}_{ij}(a, b) = -\mathbf{h}_{ji}(\rho(\bar{a}), \rho(\bar{b}))$. $\square$

**Lemma 4.4.** *Every element $x \in \mathfrak{stp}_m^0(R, ^-)$ can be written as*

$$x = \sum_{i \in I_x} \boldsymbol{\lambda}(a_i, b_i) + \boldsymbol{\mu}(c) + \sum_{j=2}^{m} \mathbf{h}_{1j}(1, d_j), \tag{4.10}$$

*where $I_x$ is a finite index set, $a_i, b_i, c, d_j \in R$ for $i \in I_x$ and $j = 2, \ldots, m$. Moreover,*

$$\boldsymbol{\mu}([a, b]) = \boldsymbol{\lambda}(a, b) + \boldsymbol{\lambda}(\rho(\bar{a}), \rho(\bar{b})) \tag{4.11}$$

*for homogeneous $a, b \in R$.*

*Proof.* Recall that $\mathfrak{stp}_m^0(R, ^-)$ is spanned by $\mathbf{h}_{ij}(a, b)$ for homogeneous $a, b \in R$ and $1 \leqslant i \neq j \leqslant m$. It suffices to show that every $\mathbf{h}_{ij}(a, b)$ can be written in the form of (4.10).

We first observe that

$$-\mathbf{h}_{i1}(\rho(\bar{a}), \rho(\bar{b})) = \mathbf{h}_{1i}(a, b) = \mathbf{h}(a, b) + (-1)^{|a||b|}\mathbf{h}_{1i}(1, ba)$$

for $a, b \in R$ and $i = 2, \ldots, m$.

If $2 \leqslant i \neq j \leqslant m$, then

$$\begin{aligned}
\mathbf{h}_{ij}(a, b) &= [\mathbf{f}_{ij}(a), \mathbf{g}_{ji}(b)] \\
&= [[\mathbf{t}_{i1}(1), \mathbf{f}_{1j}(a)], \mathbf{g}_{ji}(b)] \\
&= [\mathbf{f}_{1j}(a), \mathbf{g}_{j1}(b)] + [\mathbf{t}_{i1}(1), \mathbf{t}_{1i}(ab)] \\
&= \mathbf{h}_{1j}(a, b) + [\mathbf{t}_{i1}(1), \mathbf{t}_{1i}(ab)] \\
&= \mathbf{h}_{1j}(a, b) + \mathbf{h}_{ij}(1, ab) - \mathbf{h}_{1j}(1, ab) \\
&= \mathbf{h}_{1j}(a, b) - \boldsymbol{\mu}(ab) + \mathbf{h}_{i1}(1, ab),
\end{aligned}$$

which is of the form (4.10) since $\mathbf{h}_{1j}(a, b)$ and $\mathbf{h}_{i1}(1, ab)$ have already been of the form (4.10).

Now, we prove the equality (4.11). For $2 \leqslant i \neq j \leqslant m$, we have already obtained that

$$\begin{aligned}
\boldsymbol{\mu}(ab) &= \mathbf{h}_{1j}(a, b) + \mathbf{h}_{i1}(1, ab) - \mathbf{h}_{ij}(a, b) \\
&= \mathbf{h}_{1j}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1i}(1, \rho(\bar{b})\rho(\bar{a})) - \mathbf{h}_{ij}(a, b).
\end{aligned}$$

It follows from Lemma 4.3 that

$$\begin{aligned}
\boldsymbol{\mu}(ba) &= -(-1)^{|a||b|}\boldsymbol{\mu}(\rho(\bar{a})\rho(\bar{b})) \\
&= -(-1)^{|a||b|}(\mathbf{h}_{1i}(\rho(\bar{a}), \rho(\bar{b})) - (-1)^{|a||b|}\mathbf{h}_{1j}(1, ba) - \mathbf{h}_{ji}(\rho(\bar{a}), \rho(\bar{b}))) \\
&= -(-1)^{|a||b|}\mathbf{h}_{1i}(\rho(\bar{a}), \rho(\bar{b})) + \mathbf{h}_{1j}(1, ba) - (-1)^{|a||b|}\mathbf{h}_{ij}(a, b).
\end{aligned}$$

Hence,

$$\boldsymbol{\mu}([a, b]) = \boldsymbol{\mu}(ab) - (-1)^{|a||b|}\boldsymbol{\mu}(ba)$$

$$
\begin{aligned}
&= \mathbf{h}_{1j}(a,b) - (-1)^{|a||b|}\mathbf{h}_{1j}(1,ba) \\
&\quad + \mathbf{h}_{1i}(\rho(\bar{a}),\rho(\bar{b})) - (-1)^{|a||b|}\mathbf{h}_{1i}(1,\rho(\bar{b})\rho(\bar{a})) \\
&= \boldsymbol{\lambda}(a,b) + \boldsymbol{\lambda}(\rho(\bar{a}),\rho(\bar{b})).
\end{aligned}
$$

This completes the proof. $\square$

Now, we may proceed to prove Proposition 4.1,

*Proof of Proposition 4.1.* Recall (4.1) that

$$
{}_{+}\mathrm{HD}_1(R,{}^{-}\circ\rho) = \left\{ \sum_i \langle a_i,b_i\rangle \in \langle R,R\rangle \,\middle|\, \sum_i \overline{[a_i,b_i]} = -\sum_i [\rho(a_i),\rho(b_i)] \right\},
$$

where $\langle R,R\rangle = (R\otimes_{\Bbbk} R)/I$ and $I$ is the $\Bbbk$-submodule of $R\otimes_{\Bbbk} R$ spanned by $a\otimes b + (-1)^{|a||b|} b\otimes a$, $a\otimes b + \rho(\bar{a})\otimes\rho(\bar{b})$ and $(-1)^{|a||c|}ab\otimes c + (-1)^{|b||a|}bc\otimes a + (-1)^{|c||b|}ca\otimes b$ for homogeneous $a,b,c\in R$.

By Lemmas 4.3 and 4.4, there exists a well-defined $\Bbbk$–linear map

$$
\begin{aligned}
&\eta : \langle R,R\rangle \to \mathfrak{stp}_m(R,{}^{-}), \\
&\langle a,b\rangle \mapsto \boldsymbol{\lambda}(a,b) - \frac{1}{2}\boldsymbol{\mu}([a,b]) = \frac{1}{2}(\boldsymbol{\lambda}(a,b) - \boldsymbol{\lambda}(\rho(\bar{a}),\rho(\bar{b}))).
\end{aligned}
$$

We will prove that its restriction on ${}_{+}\mathrm{HD}_1(R,{}^{-}\circ\rho)$ is an isomorphism of $\Bbbk$-modules onto $\ker\psi$.

We claim that $\eta\left({}_{+}\mathrm{HD}_1(R,{}^{-}\circ\rho)\right) \subseteq \ker\psi$. For $\sum_i\langle a_i,b_i\rangle \in {}_{+}\mathrm{HD}_1(R,{}^{-}\circ\rho)$, we have

$$
\sum_i \overline{[a_i,b_i]} = -\sum_i [\rho(a_i),\rho(b_i)].
$$

Hence,

$$
\psi(\eta(\textstyle\sum_i\langle a_i,b_i\rangle)) = \frac{1}{2}\sum_i \psi(\boldsymbol{\lambda}(a_i,b_i) - \boldsymbol{\lambda}(\rho(\bar{a}_i),\rho(\bar{b}_i))) = \frac{1}{2}\sum_i e_{11}([a_i,b_i] - [\rho(\bar{a}_i),\rho(\bar{b}_i)]) = 0.
$$

Conversely, let $\boldsymbol{x}\in\ker\psi\subseteq\mathfrak{stp}_m^0(R,{}^{-})$ (see Proposition 3.4). It follows from Lemma 4.4 that

$$
\boldsymbol{x} = \sum_{i\in I_{\boldsymbol{x}}} \boldsymbol{\lambda}(a_i,b_i) + \boldsymbol{\mu}(c) + \sum_{j=2}^{m} \mathbf{h}_{1j}(1,d_j),
$$

where $I_{\boldsymbol{x}}$ is a finite index set, $a_i,b_i,c,d_j\in R$ for $i\in I_{\boldsymbol{x}}$ and $j=2,\ldots,m$, and hence,

$$
0 = \psi(\boldsymbol{x}) = \sum_{i\in I_{\boldsymbol{x}}} e_{11}([a_i,b_i]) + e_{11}(c_{(-)}) + \sum_{j=2}^{m}(e_{11}(d_j) - e_{jj}(\bar{d}_j)),
$$

which implies that $d_j=0$ for $j=2,\ldots,m$ and

$$
\sum_{i\in I_{\boldsymbol{x}}}[a_i,b_i] = -c_{(-)} \in R_{(-)}.
$$

Since $\frac{1}{2}\in\Bbbk$ and $\boldsymbol{\mu}(\bar{a}) = -\boldsymbol{\mu}(\rho(a))$ for homogeneous $a\in R$, we have

$$
\boldsymbol{\mu}(c) = \frac{1}{2}\boldsymbol{\mu}(c_{(-)}) = -\frac{1}{2}\sum_{i\in I_{\boldsymbol{x}}}\boldsymbol{\mu}([a_i,b_i]).
$$

Hence, we conclude that

$$
x = \sum_{i\in I_{\boldsymbol{x}}}(\boldsymbol{\lambda}(a_i,b_i) - \frac{1}{2}\boldsymbol{\mu}([a_i,b_i])) = \sum_{i\in I_x}\eta(\langle a_i,b_i\rangle),
$$

and $\sum_{i\in I_x}\langle a_i, b_i\rangle \in {}_+\mathrm{HD}_1(R, {}^- \circ \rho)$.

It remains to show the injectivity of $\eta$. Define a $\Bbbk$–bilinear map

$$\alpha : \mathfrak{gl}_{m|m}(R) \times \mathfrak{gl}_{m|m}(R) \to \langle R, R\rangle$$

by

$$\alpha(e_{ij}(a), e_{kl}(b)) = \delta_{jk}\delta_{il}(-1)^{|i|(|i|+|a|+|b|)}\langle a, b\rangle$$

for homogeneous $a, b \in R$ and $1 \leqslant i, j \leqslant 2m$, where $|i|$ is the parity of $i$ given by (2.2). It is verified that $\alpha$ is a 2-cocycle on the Lie superalgebra $\mathfrak{gl}_{m|m}(R)$.

Now, the restriction of $\alpha$ on $\mathfrak{p}_m(R, {}^-) \times \mathfrak{p}_m(R, {}^-)$ is a 2-cocycle on $\mathfrak{p}_m(R, {}^-) \subseteq \mathfrak{gl}_{m|m}(R)$. Hence, there is a Lie superalgebra structure on $\mathfrak{p}_m(R, {}^-) \oplus \langle R, R\rangle$:

$$[x \oplus c, y \oplus c'] = [x, y] \otimes \alpha(x, y), \quad x, y \in \mathfrak{p}_m(R, {}^-) \text{ and } c, c' \in \langle R, R\rangle.$$

Observing that $t_{ij}(a) \oplus 0$, $f_{ij}(a) \oplus 0$ and $g_{ij}(a) \oplus 0 \in \mathfrak{p}_m(R, {}^-) \oplus \langle R, R\rangle$ satisfy all relations (STP00)-(STP12), there is a canonical homomorphism of Lie superalgebras

$$\phi : \mathfrak{stp}_m(R, {}^-) \to \mathfrak{p}_m(R, {}^-) \oplus \langle R, R\rangle$$

such that

$$\phi(\mathbf{t}_{ij}(a)) = t_{ij}(a) \oplus 0, \quad \phi(\mathbf{f}_{ij}(a)) = f_{ij}(a) \oplus 0, \quad \phi(\mathbf{g}_{ij}(a)) = g_{ij}(a) \oplus 0,$$

for $a \in R$ and $1 \leqslant i \neq j \leqslant m$. We now compute that

$$\begin{aligned}
\phi(\mathbf{h}_{ij}(a, b)) &= \phi([\mathbf{f}_{ij}(a), \mathbf{g}_{ji}(b)]) = [f_{ij}(a) \oplus 0, g_{ji}(b) \oplus 0] \\
&= [f_{ij}(a), g_{ji}(b)] \oplus \alpha(f_{ij}(a), g_{ji}(b)) \\
&= (t_{ii}(ab) - t_{jj}(\rho(\bar{a})\rho(\bar{b}))) \oplus (\langle a, b\rangle - \langle \rho(\bar{a}), \rho(\bar{b})\rangle) \\
&= (t_{ii}(ab) - t_{jj}(\rho(\bar{a})\rho(\bar{b}))) \oplus 2\langle a, b\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\phi(\boldsymbol{\lambda}(a, b)) &= \phi(\mathbf{h}_{1i}(a, b) - (-1)^{|a||b|}\mathbf{h}_{1i}(1, ba)) \\
&= t_{11}([a, b]) \oplus (2\langle a, b\rangle - 2(-1)^{|a||b|}\langle 1, ba\rangle).
\end{aligned}$$

Since $\langle 1, a\rangle = -\langle a, 1\rangle$ and $\langle a, 1\rangle + \langle a, 1\rangle + \langle 1, a\rangle = 0$, we obtain that $\langle 1, a\rangle = 0$. Hence,

$$\phi(\boldsymbol{\lambda}(a, b)) = t_{11}([a, b]) \oplus 2\langle a, b\rangle.$$

Since $\frac{1}{2} \in \Bbbk$,

$$\begin{aligned}
\phi(\eta(\langle a, b\rangle)) &= \frac{1}{2}(\phi(\boldsymbol{\lambda}(a, b)) - \phi(\boldsymbol{\lambda}(\rho(\bar{a}), \rho(\bar{b})))) \\
&= \frac{1}{2}(t_{11}([a, b]) \oplus 2\langle a, b\rangle_{\mathsf{d}}^{(+)} - t_{11}([\rho(\bar{a}), \rho(\bar{b})]) \oplus 2\langle \rho(\bar{a}), \rho(\bar{b})\rangle_{\mathsf{d}}^{(+)}) \\
&= \frac{1}{2}(t_{11}([a, b] - [\rho(\bar{a}), \rho(\bar{b})]) \oplus (\langle a, b\rangle_{\mathsf{d}}^{(+)} - \langle \rho(\bar{a}), \rho(\bar{b})\rangle_{\mathsf{d}}^{(+)})) \\
&= \frac{1}{2}t_{11}([a, b] - [\rho(\bar{a}), \rho(\bar{b})]) \oplus 2\langle a, b\rangle_{\mathsf{d}}^{(+)},
\end{aligned}$$

which shows that $\eta$ is injective and completes the proof. $\square$

## 5 The universality of the central extension $\psi$

It is shown in Section 3 that the canonical homomorphism $\psi : \mathfrak{stp}_m(R, ^-) \to \mathfrak{p}_m(R, ^-)$ is a central extension, whose kernel has been explicitly characterized in Section 4. In this section, we will prove that the central extension $\psi$ is universal for $m \geqslant 5$ and thus obtain a precise description of the second homology $\mathfrak{p}_m(R, ^-)$.

A necessary condition for the universality of $\psi$ is the perfectness $\mathfrak{stp}_m(R, ^-)$, which can be easily observed from the defining relations (STP03), (STP05) and (STP07). Next, we proceed to prove the universality of $\psi$.

Let $\varphi : \mathfrak{E} \to \mathfrak{p}_m(R, ^-)$ be an arbitrary central extension of $\mathfrak{p}_m(R, ^-)$ with $m \geqslant 3$. For $a \in R$ and $1 \leqslant i \neq j \leqslant m$, we pick

$$\hat{t}_{ij}(a) \in \varphi^{-1}(t_{ij}(a)), \quad \hat{f}_{ij}(a) \in \varphi^{-1}(f_{ij}(a)), \text{ and } \hat{g}_{ij}(a) \in \varphi^{-1}(g_{ij}(a)).$$

Obviously, the element $[\hat{x}, \hat{y}] \in \mathfrak{E}$ is independent the choice of the representatives $\hat{x} \in \varphi^{-1}(x)$ and $\hat{y} \in \varphi^{-1}(y)$ for $x, y \in \mathfrak{p}_m(R, ^-)$. Moreover, we have the following lemma:

**Lemma 5.1.** *In the Lie superalgebra $\mathfrak{E}$, the following equalities hold:*

*(i)* $[\hat{f}_{ik}(a), \hat{g}_{kj}(b)] = [\hat{f}_{il}(a), \hat{g}_{lj}(b)]$,

*(ii)* $[\hat{t}_{ik}(a), \hat{f}_{kj}(b)] = [\hat{t}_{il}(a), \hat{f}_{lj}(b)]$,

*(iii)* $[\hat{g}_{ik}(a), \hat{t}_{kj}(b)] = [\hat{g}_{il}(a), \hat{t}_{lj}(b)]$.

*for $a, b \in R$ and distinct $i, j, k, l$.*

*Proof.* (i) Since $[\hat{f}_{ik}(a), \hat{t}_{lk}(1)] + \hat{f}_{il}(a) \in \ker \varphi$ that is included in the center of $\mathfrak{E}$, we deduce

$$\begin{aligned} [\hat{f}_{il}(a), \hat{g}_{lj}(b)] &= -[[\hat{f}_{ik}(a), \hat{t}_{lk}(1)], \hat{g}_{lj}(b)] \\ &= -[[\hat{f}_{ik}(a), \hat{g}_{lj}(b)], \hat{t}_{lk}(1)] - [\hat{f}_{ik}(a), [\hat{t}_{lk}(1), \hat{g}_{lj}(b)]] \\ &= 0 + [\hat{f}_{ik}(a), \hat{g}_{kj}(b)], \end{aligned}$$

which shows (i). The equalities (ii) and (iii) follows similarly. □

According to Lemma 5.1, we define for each pair $(i, j)$ with $1 \leqslant i \neq j \leqslant m$ that:

$$\tilde{t}_{ij}(a) := [\hat{f}_{ik}(a), \hat{g}_{kj}(1)], \quad \tilde{f}_{ij}(a) := [\hat{t}_{ik}(1), \hat{f}_{kj}(a)], \text{ and } \tilde{g}_{ij}(a) := [\hat{g}_{ik}(a), \hat{t}_{kj}(1)], \tag{5.1}$$

where $a \in R$ and $1 \leqslant k \leqslant m$ is an arbitrary integer such that $k \neq i, j$.

**Lemma 5.2.** *Suppose $m \geqslant 3$. Let $\tilde{t}_{ij}(a), \tilde{f}_{ij}(a)$ and $\tilde{g}_{ij}(a)$ be the elements of $\mathfrak{E}$ given in (5.1), where $a \in R$ and $1 \leqslant i \neq j \leqslant m$. Then they satisfy all relations (STP00)-(STP12) except (STP04) and (STP10). Moreover, for $a, b \in R$, we have*

$$[\tilde{t}_{ik}(a), \tilde{t}_{jk}(b)] = 0, \qquad \textit{if } i, j, k \textit{ are distinct,} \tag{STP04a}$$
$$[\tilde{t}_{ij}(a), \tilde{t}_{kl}(b)] = 0, \qquad \textit{if } i, j, k, l \textit{ are distinct,} \tag{STP04b}$$
$$[\tilde{g}_{ij}(a), \tilde{g}_{ij}(b)] = 0, \qquad \textit{if } i \neq j, \tag{STP10a}$$
$$[\tilde{g}_{ij}(a), \tilde{g}_{jk}(b)] = 0, \qquad \textit{if } i, j, k \textit{ are distinct.} \tag{STP10b}$$

*Proof.* The $\Bbbk$-linearity of $\tilde{t}_{ij}$ is obvious since both $\hat{f}_{ik}(ca)$ and $c\hat{f}_{ik}(a)$ are contained in $\varphi^{-1}(f_{ik}(a))$ for $a \in R$ and $c \in \Bbbk$. Similarly, we have the $\Bbbk$-linearity of $\tilde{f}_{ij}$ and $\tilde{g}_{ij}$, which shows (STP00).

For (STP01), we aim to show $\tilde{f}_{ij}(\bar{a}) = \tilde{f}_{ji}(\rho(a))$ for $a \in R$. Note that $m \geqslant 3$, we choose $1 \leqslant k \leqslant m$ such that $i, j, k$ are distinct and set

$$\tilde{h}_{ik} = [\hat{f}_{ik}(1), \hat{g}_{ki}(1)], \tag{5.2}$$

which is independent of the choice of $k$ and the representatives $\hat{f}_{ik}(1) \in \varphi^{-1}(f_{ik}(1))$ and $\hat{g}_{ki}(1) \in \varphi^{-1}(g_{ki}(1))$. Then we compute that

$$[\tilde{h}_{ik}, \tilde{f}_{ij}(a)] = [\tilde{h}_{ik}, [\hat{t}_{ik}(1), \hat{f}_{kj}(a)]] = [[\tilde{h}_{ik}, \hat{t}_{ik}(1)], \hat{f}_{kj}(a)] + [\hat{t}_{ik}(1), [\tilde{h}_{ik}, \hat{f}_{kj}(a)]].$$

Now, $\varphi(h_{ik}) = e_{ii}(1) - e_{kk}(1) - e_{m+i,m+i}(1) + e_{m+k,m+k}(1), \varphi(\hat{t}_{ik}(1)) = t_{ik}(1)$ and $\varphi(\hat{f}_{kj}(a)) = f_{kj}(a)$. Hence,

$$[\tilde{h}_{ik}, \hat{t}_{ik}(1)] \in \varphi^{-1}(2t_{ik}(1)), \text{ and } [\tilde{h}_{ik}, \hat{f}_{kj}(a)] \in \varphi^{-1}(-f_{kj}(a)).$$

It yields that $[\tilde{h}_{ik}, \tilde{f}_{ij}(a)] = \tilde{f}_{ij}(a)$.

Observing that $f_{ij}(\bar{a}) = f_{ji}(\rho(a))$ in $\mathfrak{p}_m(R, {}^-)$, we have $\tilde{f}_{ij}(\bar{a}) - \tilde{f}_{ji}(\rho(a)) \in \ker\varphi$ which is contained in the center of $\mathfrak{E}$. Applying $\tilde{h}_{ik}$, we obtain that

$$0 = [\tilde{h}_{ik}, \tilde{f}_{ij}(\bar{a}) - \tilde{f}_{ji}(\rho(a))] = \tilde{f}_{ij}(\bar{a}) - \tilde{f}_{ji}(\rho(a)).$$

This shows $\tilde{f}_{ij}(a)$ satisfies the relation (STP01). All other relations can be verified similarly. □

*Remark* 5.3. The relation (STP04) states that

$$[\mathbf{t}_{ij}(a), \mathbf{t}_{kl}(b)] = 0, \text{ if } i \neq j \neq k \neq l \neq i,$$

which is equivalent to (STP04a), (STP04b) and

$$[\mathbf{t}_{ij}(a), \mathbf{t}_{ik}(b)] = 0, \text{ if } i, j, k \text{ are distinct.} \tag{STP04c}$$

By Lemma 5.2, the elements $\tilde{t}_{ij}(a)$'s in an arbitrary central extension $\mathfrak{E}$ of $\mathfrak{p}_m(R, {}^-)$ always satisfy (STP04a) and (STP04b), but do not necessarily satisfy (STP04c). Such examples will appear in central extensions of $\mathfrak{p}_3(R, {}^-)$. The similar phenomenon also occurs for the relation (STP10).

**Proposition 5.4.** *Let $(R, {}^-)$ be a unital associative superalgebra with superinvolution and $m \geqslant 5$. Then $\psi : \mathfrak{stp}_m(R, {}^-) \to \mathfrak{p}_m(R, {}^-)$ is a universal central extension.*

*Proof.* Let $\varphi : \mathfrak{E} \to \mathfrak{p}_m(R, {}^-)$ be an arbitrary central extension. Take $\tilde{t}_{ij}(a), \tilde{f}_{ij}(a), \tilde{g}_{ij}(a) \in \mathfrak{E}$ for $a \in R$ and $1 \leqslant i \neq j \leqslant m$ as in (5.1). Then we have already known from Lemma 5.2 that they satisfy all relations (STP00)-(STP12) except (STP04) and (STP10). Now, under the additional assumption that $m \geqslant 5$, we will show that these elements also satisfy (STP04) and (STP10).

For (STP04), since (STP04a) and (STP04b) have already been verified in Lemma 5.2, it suffices to show

$$[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = 0, \text{ if } i, j, k \text{ are distinct.} \tag{STP04c}$$

Indeed, we observe that $[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] \in \ker\varphi$. Since $m \geqslant 5$, we are allowed to choose $1 \leqslant l \leqslant m$ such that $l \neq i, j, k$. Applying $\tilde{h}_{lj}$ defined in (5.2), we obtain that

$$\begin{aligned} 0 &= [\tilde{h}_{lj}, [\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)]] \\ &= [[\tilde{h}_{lj}, \tilde{t}_{ij}(a)], \tilde{t}_{ik}(b)] + [\tilde{t}_{ij}(a), [\tilde{h}_{lj}, \tilde{t}_{ik}(b)]] \\ &= [\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)]. \end{aligned}$$

Then (STP04c) follows.

For (STP10), we have obtained (STP10a) and (STP10b) in Lemma 5.2. It suffices to show

$$[\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] = 0, \text{ if } i, j, k, l \text{ are distinct.} \tag{STP10c}$$

Since $m \geqslant 5$, we are permitted to choose $k'$ such $k' \neq i, j, k, l$. Hence,

$$\begin{aligned} [\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] &= [\tilde{g}_{ij}(a), [\hat{g}_{kk'}(b), \hat{t}_{k'l}(1)]] \\ &= [[\tilde{g}_{ij}(a), \hat{g}_{kk'}(b)], \hat{t}_{k'l}(1)] + (-1)^{(1+|a|)(1+|b|)}[\hat{g}_{kk'}(b), [\tilde{g}_{ij}(a), \hat{t}_{k'l}(1)]] = 0. \end{aligned}$$

This proves (STP10).

In summary, we have shown that the elements $\tilde{t}_{ij}(a), \tilde{f}_{ij}(a), \tilde{g}_{ij}(a) \in \mathfrak{E}$ with $a \in R$ and $1 \leqslant i \neq j \leqslant m$ satisfy all relations (STP00)-(STP12). Hence, there is a homomorphism of Lie superalgebras

$$\varphi' : \mathfrak{stp}_m(R, ^-) \to \mathfrak{E}$$

such that

$$\varphi'(\mathbf{t}_{ij}(a)) = \tilde{t}_{ij}(a), \quad \varphi'(\mathbf{f}_{ij}(a)) = \tilde{f}_{ij}(a), \quad \varphi'(\mathbf{g}_{ij}(a)) = \tilde{g}_{ij}(a),$$

for $a \in R$ and $1 \leqslant i \neq j \leqslant m$, i.e., $\varphi \circ \varphi' = \psi$.

To show the uniqueness of $\varphi'$, we let $\tilde{\varphi}' : \mathfrak{stp}_m(R, ^-) \to \mathfrak{E}$ be another homomorphism of Lie superalgebras such that $\varphi \circ \tilde{\varphi}' = \psi$. Then

$$\tilde{\varphi}'(\mathbf{t}_{ij}(a)) \in \varphi^{-1}(t_{ij}(a)), \quad \tilde{\varphi}'(\mathbf{f}_{ij}(a)) \in \varphi^{-1}(f_{ij}(a)), \text{ and } \tilde{\varphi}'(\mathbf{g}_{ij}(a)) \in \varphi^{-1}(g_{ij}(a)),$$

for $1 \leqslant i \neq j \leqslant m$ and homogeneous $a \in R$. Note that

$$\mathbf{t}_{ij}(a) = [\mathbf{t}_{ik}(a), \mathbf{t}_{kj}(1)], \quad \mathbf{f}_{ij}(a) = [\mathbf{t}_{ik}(1), \mathbf{g}_{kj}(a)], \text{ and } \mathbf{g}_{ij}(a) = [\mathbf{g}_{ik}(a), \mathbf{t}_{kj}(1)],$$

for $a \in R$ and distinct $i, j, k$, we deduce that

$$\tilde{\varphi}'(\mathbf{t}_{ij}(a)) = \tilde{t}_{ij}(a), \quad \tilde{\varphi}'(\mathbf{f}_{ij}(a)) = \tilde{f}_{ij}(a), \text{ and } \tilde{\varphi}'(\mathbf{g}_{ij}(a)) = \tilde{g}_{ij}(a).$$

It yields that $\tilde{\varphi}' = \varphi'$ since $\mathfrak{stp}_m(R, ^-)$ is generated by $\mathbf{t}_{ij}(a), \mathbf{f}_{ij}(a), \mathbf{g}_{ij}(a)$ with $a \in R$ and $1 \leqslant i \neq j \leqslant m$. Thus, there is a unique homomorphism $\varphi' : \mathfrak{stp}_m(R, ^-) \to \mathfrak{E}$ such that $\varphi \circ \varphi' = \psi$. Therefore, the central extension $\psi : \mathfrak{stp}_m(R, ^-) \to \mathfrak{p}_m(R, ^-)$ is universal. $\square$

**Theorem 5.5.** *Let $(R, ^-)$ be a unital associative superalgebra with superinvolution and $m \geqslant 5$. Then*

$$\mathrm{H}_2(\mathfrak{p}_m(R, ^-)) = {}_+\mathrm{HD}_1(R, ^- \circ \rho)$$

*where $\rho$ is the $\Bbbk$-linear map given in (2.5).*

*Proof.* The second homology of $\mathfrak{p}_m(R, ^-)$ can be identified with the kernel of its universal central extension $\psi$, which has been shown in Proposition 4.1 to be ${}_+\mathrm{HD}_1(R, ^- \circ \rho)$. $\square$

*Remark* 5.6. If $R$ is super-commutative, then one deduce from the definition that ${}_+\mathrm{HD}_1(R, \mathrm{id}) = 0$. Hence,

$$\mathrm{H}_2(\mathfrak{p}_m(\Bbbk) \otimes_\Bbbk R) \cong \mathrm{H}_2(\mathfrak{p}_m(R, \rho)) \cong {}_+\mathrm{HD}_1(R, \mathrm{id}) = 0.$$

This recovers the results about the second homology of $\mathfrak{p}_m(\Bbbk) \otimes_\Bbbk R$ given in [8] and [14].

In the special case where $(R, ^-) = (S \oplus S^{\mathrm{op}}, \mathrm{ex})$ for a unital associative superalgebra $S$, we have

**Corollary 5.7.** *Let $S$ be an arbitrary unital associative superalgebra and $m \geqslant 5$. Then*

$$\mathrm{H}_2(\mathfrak{sl}_{m|m}(S)) \cong {}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex} \circ \rho) \cong \mathrm{HC}_1(S),$$

*where $\mathrm{HC}_1(S)$ is the first $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology of $S$ as defined in [4].*

*Proof.* By Example 2.1, the Lie superalgebra $\mathfrak{sl}_{m|m}(S)$ is isomorphic to $\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})$. Theorem 5.5 insures that

$$\mathrm{H}_2(\mathfrak{sl}_{m|m}(S)) \cong \mathrm{H}_2(\mathfrak{p}_m(S \oplus S^{\mathrm{op}}, \mathrm{ex})) \cong {}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex} \circ \rho),$$

for $m \geqslant 5$.

Next, we identify ${}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex} \circ \rho)$ with the first $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology $\mathrm{HC}_1(S)$ defined in [4]. In the $\Bbbk$-module $\langle S \oplus S^{\mathrm{op}}, S \oplus S^{\mathrm{op}} \rangle$ defined in (4.1), we have

$$\langle a \oplus 0, 0 \oplus b \rangle = \langle (a \oplus 0)(1 \oplus 0), 0 \oplus b \rangle + 0 + 0$$

$$
\begin{aligned}
&=\langle (a\oplus 0)(1\oplus 0), 0\oplus b\rangle + (-1)^{|a||b|}\langle (1\oplus 0)(0\oplus b), a\oplus 0\rangle\\
&\quad + (-1)^{|a||b|}\langle (0\oplus b)(a\oplus 0), 1\oplus 0\rangle\\
&=0.
\end{aligned}
$$

This shows that

$$\langle a_1\oplus a_2, b_1\oplus b_2\rangle = \langle a_1\oplus 0, b_1\oplus 0\rangle + \langle a_2\oplus 0, b_2\oplus 0\rangle,$$

for $a_1, a_2, b_1, b_2 \in S$.

Let $I_{\mathsf{c}}$ be the $\Bbbk$-submodule of $S$ generated by $a\otimes b - (-1)^{|a||b|} b\otimes a$ and $(-1)^{|a||c|} ab\otimes c + (-1)^{|b||a|} bc\otimes a + (-1)^{|c||b|} ca\otimes b$ for homogeneous $a, b, c \in S$ and $\langle S, S\rangle_{\mathsf{c}} = (S\otimes S)/I_{\mathsf{c}}$. Then one observes that

$$\langle a_1\oplus a_2, b_1\oplus b_2\rangle \mapsto \langle a_1, b_1\rangle_{\mathsf{c}} + \langle a_2, b_2\rangle_{\mathsf{c}}$$

defines an isomorphism $\langle S\oplus S^{\mathrm{op}}, S\oplus S^{\mathrm{op}}\rangle \to \langle S, S\rangle_{\mathsf{c}}$. Its restriction on ${}_{+}\mathrm{HD}_1(S\oplus S^{\mathrm{op}}, \mathrm{ex}\circ\rho)$ gives an isomorphism onto

$$\mathrm{HC}_1(S) := \left\{ \sum_i \langle a_i, b_i\rangle_{\mathsf{c}} \in \langle S, S\rangle_{\mathsf{c}} \,\middle|\, \sum_i [a_i, b_i] = 0 \right\}.$$

This completes the proof. □

*Remark* 5.8. The above corollary recovers the second homology of $\mathfrak{sl}_{m|m}(S)$ for $m \geqslant 5$ obtained in [4]. As a byproduct, we obtain the isomorphism

$${}_{+}\mathrm{HD}_1(S\oplus S^{\mathrm{op}}, \mathrm{ex}\circ\rho) \cong \mathrm{HC}_1(S),$$

which indicates that the first $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology can be regarded as a special case of the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology. However, it is unknown yet whether such an isomorphism exists for higher degree cyclic homology and higher degree dihedral homology.

# 6 The second homology of $\mathfrak{p}_4(R, ^-)$

The second homology of $\mathfrak{p}_m(R, ^-)$ for $m \geqslant 5$ have been explicitly characterized in the previous section. However, the central extension $\psi : \mathfrak{stp}_4(R, ^-) \to \mathfrak{p}_4(R, ^-)$ is not necessarily universal. This section is devoted to explicitly constructing a universal central extension of $\mathfrak{stp}_4(R, ^-)$, which can be accomplished by creating a 2-cocycle on $\mathfrak{stp}_4(R, ^-)$.

Such a 2-cocycle takes values in the $\Bbbk$-module $R/(R_{(-)}\cdot R)$, where $R_{(-)}\cdot R$ is the right ideal of $R$ generated by $\bar{a} - \rho(a)$ for $a \in R$. In fact, the $\Bbbk$-module $R/(R_{(-)}\cdot R)$ is a super-commutative $\Bbbk$-superalgebra since $[R, R]\cdot R \subseteq R_{(-)}\cdot R$. We denote $\boldsymbol{\pi} : R \to R/(R_{(-)}\cdot R)$ the canonical quotient map of $\Bbbk$–modules. It satisfies

$$\boldsymbol{\pi}(\bar{a}b) = \boldsymbol{\pi}(\rho(a)b), \quad a, b \in R. \tag{6.1}$$

Similar to Proposition 3.3, $\mathfrak{stp}_4(R, ^-)$ is decomposed as a direct sum of Lie sub-superalgebras:

$$\mathfrak{stp}_4(R, ^-) = \mathfrak{a}\oplus\mathfrak{b}$$

where

$$
\begin{aligned}
\mathfrak{a} &:= \mathrm{span}_{\Bbbk}\{\mathbf{h}_{ij}(a,b), \mathbf{t}_{ij}(a), \mathbf{f}_i(a), \mathbf{f}_{ij}(a) | a, b \in R, 1 \leqslant i \neq j \leqslant 4\},\\
\mathfrak{b} &:= \mathrm{span}_{\Bbbk}\{\mathbf{g}_i(a), \mathbf{g}_{ij}(a) | a \in R, 1 \leqslant i \neq j \leqslant 4\}.
\end{aligned}
$$

Then, we define a $\Bbbk$-linear map $\beta_0 : \mathfrak{b}\times\mathfrak{b} \to R/(R_{(-)}\cdot R)$ by

$$
\begin{aligned}
\beta_0(\mathbf{g}_{ij}(a), \mathbf{g}_{kl}(b)) &= \epsilon(ijkl)\boldsymbol{\pi}(a\cdot\rho(b)),\\
\beta_0(\mathbf{g}_i(a), \mathfrak{b}) &= \beta_0(\mathfrak{b}, \mathbf{g}_i(a)) = 0,
\end{aligned}
$$

for $a, b \in R$, $1 \leqslant i \neq j \leqslant 4$ and $1 \leqslant k \neq l \leqslant 4$, where $\epsilon(ijkl)$ denotes the sign of the permutation $(ijkl)$ if $(ijkl)$ is a permutation of $\{1, 2, 3, 4\}$ and denotes 0 if $(ijkl)$ is not a permutation. Such a $\mathbb{k}$-linear map $\beta_0$ is well-defined since $\mathbf{g}_{ij}(\bar{a}) = -\mathbf{g}_{ji}(\rho(a))$ while $\boldsymbol{\pi}(ab) = \boldsymbol{\pi}(\rho(a)b)$.

Furthermore, the $\mathbb{k}$–bilinear map $\beta_0 : \mathfrak{b} \times \mathfrak{b} \to R/(R_{(-)} \cdot R)$ is extended to a $\mathbb{k}$–bilinear map

$$\beta : \mathfrak{stp}_4(R, {}^-) \times \mathfrak{stp}_4(R, {}^-) \to R/(R_{(-)} \cdot R)$$

such that $\mathfrak{a}$ lies in the radical of $\beta$, i.e.,

$$\beta(\mathfrak{a}, \mathfrak{stp}_4(R, {}^-)) = \beta(\mathfrak{stp}_4(R, {}^-), \mathfrak{a}) = 0.$$

Now, we can show that:

**Lemma 6.1.** *The $\mathbb{k}$-bilinear map $\beta$ is a 2-cocycle on $\mathfrak{stp}_4(R, {}^-)$ with values in $R/(R_{(-)} \cdot R)$.*

*Proof.* We have to show $\beta$ satisfies

$$\beta(y, x) = -(-1)^{|x||y|}\beta(x, y), \tag{6.2}$$

$$(-1)^{|x||z|}\beta([x, z], y) + (-1)^{|y||x|}\beta([y, z], x) + (-1)^{|z||y|}\beta([z, x], y) = 0, \tag{6.3}$$

for homogeneous $x, y, z \in \mathfrak{stp}_4(R, {}^-)$.

For (6.2), it suffices to show

$$\beta(\mathbf{g}_{ij}(a), \mathbf{g}_{kl}(b)) = -(-1)^{(1+|a|)(1+|b|)}\beta(\mathbf{g}_{kl}(b), \mathbf{g}_{ij}(a)),$$

for homogeneous $a, b \in R$, $i \neq j$ and $k \neq l$. Note that $\boldsymbol{\pi}(ab) = (-1)^{|a||b|}\boldsymbol{\pi}(ba)$, we deduce that

$$\begin{aligned}\beta(\mathbf{g}_{ij}(a), \mathbf{g}_{kl}(b)) &= \epsilon(ijkl)\boldsymbol{\pi}(a \cdot \rho(b)) = \epsilon(klij)(-1)^{|a||b|}\boldsymbol{\pi}(\rho(b) \cdot a)\\ &= (-1)^{|a||b|}\beta(\mathbf{g}_{kl}(\rho(b)), \mathbf{g}_{ij}(\rho(a)))\\ &= -(-1)^{(1+|a|)(1+|b|)}\beta(\mathbf{g}_{kl}(b), \mathbf{g}_{ij}(a)).\end{aligned}$$

Next, we show (6.3). Observing that $\mathfrak{a}$ is a Lie sub-superalgebra of $\mathfrak{stp}_4(R, {}^-)$ included in the radical of $\beta$ and $[\mathfrak{b}, \mathfrak{b}] = 0$, we deduce that $\beta([x, y], z) = \beta([y, z], x) = \beta([z, x], y) = 0$ if $(x, y, z)$ is contained in one of the subspaces $\mathfrak{a} \times \mathfrak{a} \times \mathfrak{a}$, $\mathfrak{a} \times \mathfrak{a} \times \mathfrak{b}$, $\mathfrak{b} \times \mathfrak{b} \times \mathfrak{b}$. Note also that (6.3) is symmtric with respect to all permutations on $\{x, y, z\}$, the proof is reduced to verify (6.3) for $x \in \mathfrak{a}$ and $y, z \in \mathfrak{b}$. In this situation, $[y, z] = 0$ and (6.3) is equivalent to

$$\beta([y, x], z) = (-1)^{|x||y|+|y||z|+|z||x|}\beta([z, x], y) \tag{6.4}$$

If $x = \boldsymbol{f}_{ij}(a)$ or $x = \boldsymbol{f}_i(a)$, then $[x, \mathfrak{b}] \subseteq \mathfrak{a}$ is included in the radical of $\beta$. It yields that both sides of (6.4) are zero, and hence, we may assume that $x = \boldsymbol{h}_{ij}(a)$ or $x = \boldsymbol{t}_{ij}(a)$. In this situation, it is also obvious that both sides of (6.4) are zero if $y = \boldsymbol{g}_i(a)$ or $z = \boldsymbol{g}_i(a)$. Now, it remains to verify the following two equalities

$$\beta([\mathbf{g}_{ij}(a), \mathbf{t}_{rs}(c)], \mathbf{g}_{kl}(b)) = -(-1)^{(|a|+|b|)(1+|c|)+|a||b|}\beta([\mathbf{g}_{kl}(b), \mathbf{t}_{rs}(c)], \mathbf{g}_{ij}(a)), \tag{6.5}$$

$$\beta([\mathbf{g}_{ij}(a), \mathbf{h}_{rs}(c, c')], \mathbf{g}_{kl}(b)) = -(-1)^{(|a|+|b|)(1+|c|+|c'|)+|a||b|}\beta([\mathbf{g}_{kl}(b), \mathbf{h}_{rs}(c, c')], \mathbf{g}_{ij}(a)), \tag{6.6}$$

for homogeneous $a, b, c, c' \in R$ and $i \neq j$, $k \neq l$ and $r \neq s$.

For (6.5), we compute that

$$\begin{aligned}\beta([\boldsymbol{g}_{ij}(a), \boldsymbol{t}_{rs}(c)], \boldsymbol{g}_{kl}(b)) &= \delta_{jr}\beta(\boldsymbol{g}_{is}(ac), \boldsymbol{g}_{kl}(b)) - \delta_{ir}\beta(\boldsymbol{g}_{js}(\rho(\bar{a})c), \boldsymbol{g}_{kl}(b))\\ &= \delta_{jr}\epsilon(iskl)\boldsymbol{\pi}(ac\rho(b)) - \delta_{ir}\epsilon(jskl)\boldsymbol{\pi}(\rho(\bar{a})c\rho(b))\\ &= (\delta_{jr}\epsilon(iskl) - \delta_{ir}\epsilon(jskl))\boldsymbol{\pi}(ac\rho(b)).\end{aligned}$$

Then, (6.5) follows from the facts that $\boldsymbol{\pi}(abc) = (-1)^{|a||b|+|b||c|+|c||a|}\boldsymbol{\pi}(cba)$ and

$$\delta_{jr}\epsilon(iskl) - \delta_{ir}\epsilon(jskl) = \delta_{lr}\epsilon(ksij) - \delta_{kr}\epsilon(lsij)$$

for all $1 \leqslant i, j, k, l \leqslant 4$. The equality (6.6) follows similarly. $\square$

The 2-cocycle $\beta : \mathfrak{stp}_4(R, ^-) \times \mathfrak{stp}_4(R, ^-) \to R/(R_{(-)} \cdot R)$ gives rise to a new Lie superalgebra

$$\widehat{\mathfrak{stp}}_4(R, ^-) := \mathfrak{stp}_4(R, ^-) \oplus (R/(R_{(-)} \cdot R)),$$

under the super-bracket

$$[x \oplus c, y \oplus c] := [x, y] \oplus \beta(x, y)$$

for $x, y \in \mathfrak{stp}_4(R, ^-)$ and $c, c' \in R/(R_{(-)} \cdot R)$, which is a central extension with the canonical projection $\psi'_4 : \widehat{\mathfrak{stp}}_4(R, ^-) \to \mathfrak{stp}_4(R, ^-)$. Furthermore, we may show that

**Proposition 6.2.** *Let $(R, ^-)$ be a unital associative superalgebra with superinvolution. Then the central extension $\psi' : \widehat{\mathfrak{stp}}_4(R, ^-) \to \mathfrak{stp}_4(R, ^-)$ is universal.*

*Proof.* We have already known from Proposition 3.4 that $\psi : \mathfrak{stp}_4(R, ^-) \to \mathfrak{p}_4(R, ^-)$ is a central extension. Hence, $\psi \circ \psi' : \widehat{\mathfrak{stp}}_4(R, ^-) \to \mathfrak{p}_4(R, ^-)$ is a central extension. It suffices to show that $\psi \circ \psi'$ is universal.

Let $\varphi : \mathfrak{E} \to \mathfrak{p}_4(R, ^-)$ be an arbitrary central extension of $\mathfrak{p}_4(R, ^-)$. We also take $\tilde{t}_{ij}(a)$, $\tilde{f}_{ij}(a)$ and $\tilde{g}_{ij}(a) \in \mathfrak{E}$ as in (5.1). By Lemma 5.2, these elements satisfy (STP00)-(STP12) except (STP04) and (STP10). While the same argument as in Theorem 5.4 also shows that (STP04) holds.

For $a \in R$, we define

$$\tilde{\pi}(a) := [\tilde{g}_{12}(a), \tilde{g}_{34}(1)] \in \mathfrak{E},$$

which is contained in the center of $\mathfrak{E}$ since $\varphi(\tilde{\pi}(a)) = 0$. We next prove that $\tilde{\pi}(R_{(-)} \cdot R) = 0$.

Let $a, b \in R$ be homogenous. We compute that

$$\begin{aligned}
\tilde{\pi}(ab) &= [\tilde{g}_{12}(ab), \tilde{g}_{34}(1)] = [[\tilde{g}_{13}(a), \tilde{t}_{32}(b)], \tilde{g}_{34}(1)] \\
&= [\tilde{g}_{13}(a), [\tilde{t}_{32}(b), \tilde{g}_{34}(1)]] = -[\tilde{g}_{13}(a), \tilde{g}_{24}(\bar{b})] \\
&= -[[\tilde{g}_{14}(1), \tilde{t}_{43}(a)], \tilde{g}_{24}(\bar{b})] = -[\tilde{g}_{14}(1), [\tilde{t}_{43}(a), \tilde{g}_{24}(\bar{b})]] \\
&= (-1)^{|a|(1+|b|)}[\tilde{g}_{14}(1), \tilde{g}_{23}(\bar{b}a)] = (-1)^{|a|(1+|b|)}[\tilde{g}_{14}(1), [\tilde{g}_{21}(\bar{b}a), \tilde{t}_{13}(1)]] \\
&= (-1)^{|a|(1+|b|)}(-1)^{1+|a|+|b|}[\tilde{g}_{21}(\bar{b}a), [\tilde{g}_{14}(1), \tilde{t}_{13}(1)]] = -(-1)^{|b|+|a||b|}[\tilde{g}_{21}(\bar{b}a), \tilde{g}_{34}(1)] \\
&= (-1)^{|a|}[\tilde{g}_{12}(\bar{a}b), \tilde{g}_{34}(1)] = (-1)^{|a|}\tilde{\pi}(\bar{a}b).
\end{aligned}$$

It follows that $(a - (-1)^{|a|}\bar{a})b \in \ker \tilde{\pi}$ for homogeneous $a, b \in R$. Hence, $\tilde{\pi}(R_{(-)} \cdot R) = 0$. We obtain a $\Bbbk$–linear map

$$R/(R_{(-)} \cdot R) \to \ker \varphi, \quad \boldsymbol{\pi}(a) \mapsto \tilde{\pi}(a).$$

Since $\tilde{\pi}(a) = [\tilde{g}_{12}(a), \tilde{g}_{34}(1)]$ and $\tilde{\pi}(\bar{a}b) = \tilde{\pi}(\rho(a)b)$, we deduce that

$$[\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] = \epsilon(ijkl)\tilde{\pi}(a \cdot \rho(b)),$$

for distinct $i, j, k, l$. Combining with (STP10a) and (STP10b), we have

$$[\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] = \epsilon(ijkl)\tilde{\pi}(a \cdot \rho(b)),$$

for $i \neq j$ and $k \neq l$.

Hence, there is a homomorphism $\varphi' : \widehat{\mathfrak{stp}}_4(R, ^-) \to \mathfrak{E}$ such that

$$\varphi'(\mathbf{t}_{ij}(a) \oplus 0) = \tilde{t}_{ij}(a), \quad \varphi'(\mathbf{f}_{ij}(a) \oplus 0) = \tilde{f}_{ij}(a), \quad \varphi'(\mathbf{g}_{ij}(a) \oplus 0) = \tilde{g}_{ij}(a), \quad \varphi'(0 \oplus \boldsymbol{\pi}(a)) = \tilde{\pi}(a),$$

where $a \in R$ and $1 \leqslant i \neq j \leqslant 4$, i.e., $\psi \circ \psi' = \varphi \circ \varphi'$. The uniqueness of $\varphi'$ follows from a similar argument as the proof of Proposition 5.4. Hence, we conclude that $\psi \circ \psi' : \widehat{\mathfrak{stp}}_4(R, ^-) \to \mathfrak{p}_4(R, ^-)$ is a universal central extension. $\square$

Using Propositions 3.4 and 6.2, we conclude that

**Theorem 6.3.** *Let $(R, ^-)$ be a unital associative superalgebra with superinvolution. Then*

$$\mathrm{H}_2(\mathfrak{p}_4(R, ^-)) = {}_+\mathrm{HD}_1(R, ^- \circ \rho) \oplus R/(R_{(-)} \cdot R). \quad \square$$

*Remark* 6.4. If $R$ is super-commutative, then $R_{(-)} = 0$. In this situation,

$$\mathrm{H}_2(\mathfrak{p}_4(\Bbbk) \otimes_{\Bbbk} R) \cong \mathrm{H}_2(\mathfrak{p}_4(R, \rho)) \cong {}_+\mathrm{HD}_1(R, \mathrm{id}) \oplus R \cong R,$$

which recovers the second homology of $\mathfrak{p}_4(\Bbbk) \otimes_{\Bbbk} R$ obtained in [8].

In the special case where $(R, ^-) = (S \oplus S^{\mathrm{op}}, \mathrm{ex})$, Theorem 6.3 recovers the result about the universal central extension of $\mathfrak{sl}_{4|4}(S)$ given in [4].

**Corollary 6.5.** *Let $S$ be an arbitrary unital associative superalgebra. Then*

$$\mathrm{H}_2(\mathfrak{sl}_{4|4}(S)) = \mathrm{HC}_1(S).$$

*Proof.* Recall from Example 2.1 that the Lie superalgebra $\mathfrak{sl}_{4|4}(S)$ is isomorphic to $\mathfrak{p}_4(S \oplus S^{\mathrm{op}}, \mathrm{ex})$. Hence,

$$\mathrm{H}_2(\mathfrak{sl}_{4|4}(S)) \cong \mathrm{H}_2(\mathfrak{p}_4(S \oplus S^{\mathrm{op}}, \mathrm{ex})) \cong {}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex} \circ \rho) \oplus R/(R_{(-)} \cdot R),$$

where $(R, ^-) = (S \oplus S^{\mathrm{op}}, \mathrm{ex})$. Now, $R_{(-)}$ contains a unit element $1 \oplus (-1)$, which yields that $R/(R_{(-)} \cdot R) = 0$. Hence,

$$\mathrm{H}_2(\mathfrak{sl}_{4|4}(S)) \cong {}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex} \circ \rho) \cong \mathrm{HC}_1(S),$$

where the last isomorphism follows from Corollary 5.7. $\square$

# 7 The second homology of $\mathfrak{p}_3(R, ^-)$

Analogous to Section 6, we will calculate the second homology of $\mathfrak{p}_3(R, ^-)$ via explicitly creating the universal central extension of $\mathfrak{stp}_3(R, ^-)$. This will be accomplished by introducing a 2-cocycle on $\mathfrak{stp}_3(R, ^-)$ with values in the $\Bbbk$-module:

$$\mathfrak{z} := \frac{R}{3R + R_{(-)} \cdot R} \oplus \frac{R}{3R + R_{(-)} \cdot R} \oplus \frac{R}{3R + R_{(-)} \cdot R},$$

where $R_{(-)} \cdot R$ is the right-ideal of $R$ generated by $\bar{a} - \rho(a)$ for $a \in R$.

Let $\boldsymbol{\pi}_i(a), i = 1, 2, 3$ denote the canonical image of $a$ in one of the three direct summands, respectively. For distinct $i, j, k$, we will also use $\epsilon(ijk)$ to denote the sign of the permutation $(ijk)$.

Recall from Lemma 3.3 that $\mathfrak{stp}_3(R, ^-)$ is spanned as a $\Bbbk$–module by

$$\mathfrak{B} := \{\mathbf{h}_{ij}(a, b), \mathbf{t}_{ij}(a), \mathbf{f}_{ij}(a), \mathbf{g}_{ij}(a), \mathbf{f}_k(a), \mathbf{g}_k(a) | a, b \in R, 1 \leqslant i, j, k \leqslant 3 \text{ with } i \neq j\}.$$

We define a $\Bbbk$–bilinear map $\beta : \mathfrak{stp}_3(R, ^-) \times \mathfrak{stp}_3(R, ^-) \to \mathfrak{z}$ as follows:

$$\beta(\mathbf{t}_{ij}(a), \mathbf{t}_{ik}(b)) = \epsilon(ijk)\boldsymbol{\pi}_i(ab),$$
$$\beta(\mathbf{f}_i(a), \mathbf{g}_{jk}(b)) = -(-1)^{(1+|a|)(1+|b|)}\beta(\mathbf{g}_{jk}(b), \mathbf{f}_i(a)) = \epsilon(ijk)\boldsymbol{\pi}_i(ab),$$

where $a, b \in R$ are homogeneous and $\{i, j, k\} = \{1, 2, 3\}$. For other pairs $(x, y) \in \mathfrak{B} \times \mathfrak{B}$, we set $\beta(x, y) = 0$. The $\Bbbk$-blinear map $\beta$ is well-defined since

$$\mathbf{f}_i(\bar{a}) = \mathbf{f}_i(\rho(a)), \quad \mathbf{f}_{ij}(\bar{a}) = \mathbf{f}_{ji}(\rho(a)), \quad \mathbf{g}_i(\bar{a}) = -\mathbf{g}_i(\rho(a)), \quad \mathbf{g}_{ij}(\bar{a}) = -\mathbf{g}_{ji}(\rho(a)),$$

while $\boldsymbol{\pi}_i(\bar{a}b) = \boldsymbol{\pi}_i(\rho(a)b)$.

**Lemma 7.1.** *The $\Bbbk$–bilinear map $\beta$ is a 2-cocycle on $\mathfrak{stp}_3(R, ^-)$ with values in $\mathfrak{z}$.*

*Proof.* Since $[R,R]\cdot R\subseteq R_{(-)}\cdot R$ implies that $\boldsymbol{\pi}_i(ab)=(-1)^{|a||b|}\boldsymbol{\pi}_i(ba)$, the $\Bbbk$-bilinear map $\beta$ satisfies

$$\beta(x,y)=-(-1)^{|x||y|}\beta(y,x), \tag{7.1}$$

for homogeneous elements $x,y\in\mathfrak{stp}_3(R,{}^-)$. It suffices to show

$$J(x,y,z):=(-1)^{|x||z|}\beta([x,y],z)+(-1)^{|y||x|}\beta([y,z],x)+(-1)^{|z||x|}\beta([z,x],y)=0, \tag{7.2}$$

for $x,y,z\in\mathfrak{stp}_3(R,{}^-)$. Since (7.2) is symmetric under all permutations of $\{x,y,z\}$, we may assume $\beta([x,y],z)\neq 0$, which only occurs when $z=\mathbf{t}_{ik}(a)$, $z=\mathbf{g}_{jk}(a)$, or $z=\mathbf{f}_i(a)$.

If $z=\mathbf{t}_{ik}(a)$ for $a\in R$ and $1\leqslant i\neq k\leqslant 3$, we pick $j$ to be the unique element of $\{1,2,3\}$ such that $\{i,j,k\}=\{1,2,3\}$. Then we directly verifed that $J(x,y,z)=0$ for all possible choices of $(x,y)\in\mathfrak{B}\times\mathfrak{B}$ such that $\beta([x,y],z)\neq 0$. The pair $(x,y)$ might be one of the following pairs

$$(\mathbf{h}_{ij}(a,a'),\mathbf{t}_{ij}(b)),(\mathbf{h}_{ik}(a,a'),\mathbf{t}_{ij}(b)),(\mathbf{t}_{ik}(a),\mathbf{t}_{kj}(b)),(\mathbf{f}_{ik}(a),\mathbf{g}_{kj}(b)),(\mathbf{f}_i(a),\mathbf{g}_{ij}(b)),(\mathbf{f}_{ij}(a),\mathbf{g}_j(b)),$$

for homogeneous $a,a',b\in R$. Similarly, $J(x,y,z)=0$ when $z=\mathbf{g}_{jk}(a)$ or $z=\mathbf{f}_i(a)$. $\square$

The 2-cocycle $\beta:\mathfrak{stp}_3(R,{}^-)\times\mathfrak{stp}_3(R,{}^-)\to\mathfrak{z}$ determines a centeral extension

$$\psi_3':\mathfrak{stp}_3(R,{}^-)\oplus\mathfrak{z}\to\mathfrak{stp}_3(R,{}^-),$$

where $\psi_3'$ is the canonical projection and the super-bracket on $\mathfrak{stp}_3(R,{}^-)\oplus\mathfrak{z}$ is given by

$$[x\oplus c,y\oplus c']=[x,y]\oplus\beta(x,y),\quad x,y\in\mathfrak{stp}_3(R,{}^-),\text{ and } c,c'\in\mathfrak{z}.$$

**Proposition 7.2.** *The central extension $\psi_3':\mathfrak{stp}_3(R,{}^-)\oplus\mathfrak{z}\to\mathfrak{stp}_3(R,{}^-)$ is universal.*

*Proof.* It suffices to show the central extension $\psi\circ\psi_3':\mathfrak{stp}_3(R,{}^-)\oplus\mathfrak{z}\to\mathfrak{p}_3(R,{}^-)$ is universal.

Let $\varphi:\mathfrak{E}\to\mathfrak{p}_3(R,{}^-)$ be an arbitrary central extension of $\mathfrak{p}_3(R,{}^-)$. Pick elements $\tilde{t}_{ij}(a)$, $\tilde{f}_{ij}(a)$, $\tilde{g}_{ij}(a)\in\mathfrak{E}$ with $a\in R$ and $1\leqslant i\neq j\leqslant 3$ as in (5.1). By Lemma 5.2, they satisfying all relations (STP00)-(STP11) except (STP04) and (STP10). Moreover, since $m=3$, there are no four distinct indices $1\leqslant i,j,k,l\leqslant 3$. Hence, (STP10a) and (STP10b) imply (STP10).

For $i\in\{1,2,3\}$, there are unique $j$ and $k$ such that $(i,j,k)$ is an even permutation of $\{1,2,3\}$. We define

$$\tilde{\pi}_i(a):=[\tilde{t}_{ij}(1),\tilde{t}_{ik}(a)]\in\ker\varphi, \tag{7.3}$$

for $i=1,2,3$ and $a\in R$.

We next show that $\tilde{\pi}_i(3R+R_{(-)}\cdot R)=0$ for $i=1,2,3$. First, we take $\tilde{h}_{ij}=[\hat{f}_{ij}(1),\hat{g}_{ji}(1)]$ and deduce that

$$\begin{aligned}0&=[\tilde{h}_{ij},\tilde{\pi}_i(a)]\\&=[\tilde{h}_{ij},[\tilde{t}_{ij}(1),\tilde{t}_{ik}(a)]]\\&=[[\tilde{h}_{ij},\tilde{t}_{ij}(1)],\tilde{t}_{ik}(a)]+[\tilde{t}_{ij}(1),[\tilde{h}_{ij},\tilde{t}_{ik}(a)]]\\&=2[\tilde{t}_{ij}(1),\tilde{t}_{ik}(a)]+[\tilde{t}_{ij}(1),\tilde{t}_{ik}(a)]\\&=\tilde{\pi}_i(3a),\end{aligned}$$

where $a\in R$ and $\{i,j,k\}=\{1,2,3\}$ are chosen as in (7.3). It follows that $\tilde{\pi}_i(3R)=0$.

Now, we claim that $\tilde{\pi}_i(ab)=(-1)^{|a|}\tilde{\pi}_i(\bar{a}b)$ for $i=1,2,3$ and homogeneous $a,b\in R$. Indeed, for $i\in\{1,2,3\}$, we may pick $j,k$ as in (7.3). Then

$$\begin{aligned}\tilde{\pi}_i(ab)&=[\tilde{t}_{ij}(1),\tilde{t}_{ik}(ab)]\\&=[\tilde{t}_{ij}(1),[\tilde{f}_{ij}(a),\tilde{g}_{jk}(b)]]\\&=[[\tilde{t}_{ij}(1),\tilde{f}_{ij}(a)],\tilde{g}_{jk}(b)]+[\tilde{f}_{ij}(a),[\tilde{t}_{ij}(1),\tilde{g}_{jk}(b)]]\end{aligned}$$

$$= (-1)^{|a|}[[\tilde{t}_{ij}(1), \tilde{f}_{ji}(\bar{a})], \tilde{g}_{jk}(b)].$$

Since $[\tilde{t}_{ij}(1), \tilde{f}_{ji}(\bar{a})] = [\tilde{t}_{ik}(1), \tilde{f}_{kj}(\bar{a})] + c$ for some $c \in \ker \varphi$, we further deduce that

$$\begin{aligned}
\tilde{\pi}_i(ab) &= (-1)^{|a|}[[\tilde{t}_{ik}(1), \tilde{f}_{ki}(\bar{a})], \tilde{g}_{jk}(b)] \\
&= (-1)^{|a|+(1+|a|)(1+|b|)}[[\tilde{t}_{ik}(1), \tilde{g}_{jk}(b)], \tilde{f}_{ki}(\bar{a})] + (-1)^{|a|}[\tilde{t}_{ik}(1), [\tilde{f}_{ki}(\bar{a}), \tilde{g}_{jk}(b)]] \\
&= 0 - (-1)^{|b|}[\tilde{t}_{ik}(1), \tilde{t}_{ij}(a\bar{b})] \\
&= -(-1)^{|b|}[[\tilde{t}_{ij}(1), \tilde{t}_{jk}(1)], \tilde{t}_{ij}(a\bar{b})] \\
&= -(-1)^{|b|}[[\tilde{t}_{ij}(1), \tilde{t}_{ij}(a\bar{b})], \tilde{t}_{jk}(1)] - (-1)^{|b|}[\tilde{t}_{ij}(1), [\tilde{t}_{jk}(1), \tilde{t}_{ij}(a\bar{b})]] \\
&= (-1)^{|b|}[\tilde{t}_{ij}(1), \tilde{t}_{ik}(a\bar{b})] \\
&= (-1)^{|b|}\tilde{\pi}_i(a\bar{b}).
\end{aligned}$$

It follows that $\tilde{\pi}_i(b) = (-1)^{|b|}\tilde{\pi}^i(\bar{b})$ and

$$\begin{aligned}
\tilde{\pi}_i(ab) &= (-1)^{|a|+|b|}\tilde{\pi}_i(\overline{ab}) = (-1)^{|a|+|b|+|a||b|}\tilde{\pi}_i(\bar{b}\bar{a}) = (-1)^{|b|+|a||b|}\tilde{\pi}_i(\bar{b}a) \\
&= (-1)^{|a|+|a||b|}\tilde{\pi}_i(\overline{\bar{b}a}) = (-1)^{|a|}\tilde{\pi}_i(\bar{a}b).
\end{aligned}$$

Therefore, we conclude that $\tilde{\pi}_i(3R + R_{(-)} \cdot R) = 0$ for $i = 1, 2, 3$.

Next, we show that $[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = \epsilon(ijk)\tilde{\pi}_i(ab)$ for $\{i, j, k\} = \{1, 2, 3\}$. We first assume that the permutation taking 1 to $i$, 2 to $j$, and 3 to $k$ has positive sign. Then

$$\begin{aligned}
[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] &= [\tilde{t}_{ij}(a), [\tilde{t}_{ij}(1), \tilde{t}_{jk}(b)]] \\
&= [[\tilde{t}_{ij}(a), \tilde{t}_{ij}(1)], \tilde{t}_{jk}(b)] + [\tilde{t}_{ij}(1), [\tilde{t}_{ij}(a), \tilde{t}_{jk}(b)]] \\
&= [\tilde{t}_{ij}(1), \tilde{t}_{ik}(ab)] \\
&= \tilde{\pi}_i(ab).
\end{aligned}$$

If the permutation $(ijk)$ has negative sign, then $(ikj)$ has positive sign. We have

$$[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = -(-1)^{|a||b|}[\tilde{t}_{ik}(b), \tilde{t}_{ij}(a)] = -(-1)^{|a||b|}\tilde{\pi}_i(ba) = -\tilde{\pi}_i(ab).$$

Hence, we conclude that $[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = \epsilon(ijk)\tilde{\pi}_i(ab)$.

Therefore, there is a homomorphism of Lie superalgebras

$$\varphi' : \widehat{\mathfrak{stp}}_3(R, {}^-) \to \mathfrak{E}$$

such that

$$\varphi'(\mathbf{t}_{ij}(a)) = \tilde{t}_{ij}(a), \quad \varphi'(\mathbf{f}_{ij}(a)) = \tilde{f}_{ij}(a), \quad \varphi'(\mathbf{g}_{ij}(a)) = \tilde{g}_{ij}(a),$$

for $1 \leqslant i \neq j \leqslant 3$ and $a \in R$. Hence, $\varphi \circ \varphi' = \psi \circ \psi'$. The uniqueness of $\varphi'$ follows from the same argument as in the proof of Proposition 5.4. $\square$

Now, we conclude from Propositions 3.4 and 7.2 that:

**Theorem 7.3.** *Let $(R, {}^-)$ be a unital associative superalgebra with superinvolution. Then*

$$\mathrm{H}_2(\mathfrak{p}_3(R, {}^-)) = {}_+\mathrm{HD}_1(R, {}^- \circ \rho) \oplus \frac{R}{3R + R_{(-)} \cdot R} \oplus \frac{R}{3R + R_{(-)} \cdot R} \oplus \frac{R}{3R + R_{(-)} \cdot R}. \quad \square$$

*Remark* 7.4. If $R$ is super-commutative, then $\mathfrak{p}_3(R, \rho) \cong \mathfrak{p}_3(\Bbbk) \otimes_{\Bbbk} R$, ${}_+\mathrm{HD}_1(R, \rho \circ \rho) = 0$ and $R_{(-)} = 0$. Hence,

$$\mathrm{H}_2(\mathfrak{p}_3(\Bbbk) \otimes_{\Bbbk} R) = (R/3R) \oplus (R/3R) \oplus (R/3R),$$

which equals 0 whenever 3 is invertible in $R$. When $\Bbbk$ is a field of characteristic zero, this coincides with the second homology group of $\mathfrak{p}_3(\Bbbk) \otimes_{\Bbbk} R$ given in [8].

In the special case where $(R, ^-) = (S \oplus S^{\mathrm{op}}, \mathrm{ex})$, Theorem 7.3 recovers the second homology of $\mathfrak{sl}_{3|3}(S)$ obtained in [4].

**Corollary 7.5.** *Let $S$ be an arbitrary unital associative superalgebra. Then*

$$\mathrm{H}_2(\mathfrak{sl}_{3|3}(S)) \cong \mathrm{HC}_1(S).$$

*Proof.* It is known from Example 2.1 that $\mathfrak{sl}_{3|3}(S)$ is isomorphic to $\mathfrak{p}_3(S \oplus S^{\mathrm{op}}, \mathrm{ex})$. On the other hand, $\mathfrak{z} = 0$ since $1 \oplus (-1)$ is invertible and is contained in the right-ideal of $S \oplus S^{\mathrm{op}}$ generated by $(S \oplus S^{\mathrm{op}})_{(-)}$. Hence, $\mathrm{H}_2(\mathfrak{sl}_{3|3}(S)) \cong {}_+\mathrm{HD}_1(S \oplus S^{\mathrm{op}}, \mathrm{ex}) \cong \mathrm{HC}_1(S)$. $\square$


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